The uniqueness condition for the double pushout transformation of algebras [1]

M. Llabrés [a] F. Rosselló [a,*]

[a] *Department of Mathematics and Computer Science, University of the Balearic Islands, E-07122 Palma de Mallorca, Spain.*

Abstract

The double pushout approach to the algebraic graph transformation of hypergraphs was invented 30 years ago and it has been generalized since then to more general objects, like for instance relational systems or total and partial unary algebras. We have recently introduced the double pushout transformation of partial and partly total algebras over an arbitrary signature. In this paper we study the uniqueness condition for these rewriting formalisms, which turns out to be given by a suitable generalization to partial algebras of the well known congruence extension property for total algebras.

Key words: Double pushout algebraic transformation; partial algebra; congruence extension property

---

1 Introduction

The algebraic approach to graph transformation was introduced in the early seventies by H. Ehrig, M. Pfender and H. J. Schneider in [13]. Although this rewriting formalism was first applied to graphs, it soon showed its potential in formalizing the transformation of more general objects, like hypergraphs, relational systems, unary total and partial algebras or even objects in a topos. One of its main appeals is actually the huge number of fields where relevant applications have been given, covering practically all computer science areas.

---

[*] Corresponding author; E-mail: `cesc.rossello@uib.es`

[1] This work has been partially supported by the Spanish DGES, grant BFM2000-1113-C02-01.

Preprint submitted to Elsevier Science *17 November 2003*

One of the main approaches to algebraic transformation, and historically the first one, is the so-called *double pushout (DPO) approach.* Although the reader is probably familiar with the main features of this DPO approach, in this introduction we shall refresh its most basic language; the reader wishing more details may look up the early survey [9] or the more recent one [7].

In the DPO approach to transformation in a category $\mathcal{C}$, a *production rule* is a pair of morphisms

$$P = (L \xleftarrow{l} K \xrightarrow{r} R).$$

Such a production rule in $\mathcal{C}$ can be *applied* to an object D when there exist a morphism $m : L \to D$ and a diagram

$$\begin{tikzcd}
L \arrow[d, "m"'] & K \arrow[l, "l"'] \arrow[r, "r"] \arrow[d, "g"'] & R \arrow[d, "g'"] \\
D & B \arrow[l, "l'"] \arrow[r, "r'"'] & H
\end{tikzcd}$$

such that both squares in it are pushout squares in $\mathcal{C}$. In other words, when there exists a *pushout complement* $(B, g : K \to B, l' : B \to D)$ of l and m (i.e., an object B and two morphisms $g : K \to B$ and $l' : B \to D$ such that D, together with m and l', is a pushout of l and g) and a pushout of r and $g : K \to B$. When such a diagram exists, we say that H has been *derived* from D by the application of rule P through $m : L \to D$; the intermediate object B is called the *context object* of this derivation. A set of such production rules in a category $\mathcal{C}$, together with a start object and a class of terminal objects in $\mathcal{C}$, form a *(DPO) High-Level Replacement System*, a *HLR-System* for short [10,11].

Even when the category $\mathcal{C}$ has all binary pushouts, given a transformation rule $P = (L \overset{l}{\leftarrow} K \overset{r}{\to} R)$ and a morphism $m : L \to D$, there need not exist any object derived from D by the application of rule P through m. And when it exists, it need not be unique even up to isomorphism. The reason is that given two composable morphisms $l : K \to L$ and $m : L \to D$, there need not exist any pushout complement of them, and when it exists, it need not be unique up to isomorphism.

Thus, previous to the development of a DPO approach to the transformation in a category $\mathcal{C}$, one has to solve the following two problems:

- The *application problem*: when can a given production rule be applied through a given morphism? If $\mathcal{C}$ has all pushouts, solving this application problem amounts to finding what has been classically called a *gluing condition*: a necessary and sufficient condition on two morphisms $l : K \to L$ and $m : L \to D$ in $\mathcal{C}$ for the existence of a pushout complement of them.
- The *uniqueness problem*: which production rules are such that when they are applied to an object through a given morphism, all derived objects are

isomorphic? Since pushouts are unique up to isomorphism, it is usual to solve this problem by finding a so-called *uniqueness condition* in $\mathcal{C}$: a necessary and sufficient condition on a morphism $l : K \to L$ for the uniqueness up to isomorphism of the pushout complement (if any) of l and each morphism $m : L \to D$ (see Definition 1 below).

The answer to these questions is already known for many categories where HLR-systems have been defined. As far as the uniqueness problem goes, let us mention that the morphisms satisfying the uniqueness condition in the usual category of sets are the injective mappings, in the usual categories of graphs, hypergraphs, Petri nets, total algebras of a unary type, and even in any topos, they are the injective homomorphisms in the corresponding category, and in the usual category of partial algebras of a unary type they are the closed and injective homomorphisms: cf. [6,10,15].

During the last years, there has been a growing interest in the algebraic transformation of total and partial algebras of an arbitrary type. This interest has been mainly motivated by the need to enrich graph-like structures by means of data type information in rule-based formal specifications of software systems with complex states, and by the development of the concept of Dynamic Abstract Data Types (DADT) [12] in order to model systems with dynamic behavior.

We have recently solved the application problem in the usual categories Alg_Σ and TAlg_Σ of partial and total algebras over an arbitrary signature Σ: see [17] and [4,5], respectively. Actually, in [5] we considered not only total algebras but *partly total* algebras: partial algebras with some fixed total operations. In the last section of [17] we also stated a solution to the uniqueness problem in Alg_Σ, but space constraints forced us to give only hints to the proofs of the main results, or even no proof at all, while in [5] a solution to this uniqueness problem for partly total algebras was announced.

The goal of this paper is to give a detailed solution to this uniqueness problem in the categories of partial and partly total algebras of an arbitrary type, providing full proofs of the results announced in [17, §5], proving the claims concerning the uniqueness condition made in [5], and establishing some new results on this topic. The uniqueness condition turns out to be given by a generalization to partial algebras of the well-known congruence extension property for total algebras: we call this generalization the *minimal congruence extension property*, the mce-property for short. The congruence extension property has been thoroughly studied for total algebras, mainly in connection to varieties and quasi-varieties [2,8,19–21], but, to our knowledge, nobody has considered it so far in the partial setting. We start its study in the last section of this paper, by establishing an algebraic sufficient condition for the mce-property and by showing that, unlike what happens for total algebras [2,8], the satisfaction of the mce-property for principal congruences does not imply its satisfaction in general.

The rest of this paper is organized as follows. Sections 2 and 3 contain preliminary material: in the former we recall some basic notions and facts on partial algebras, and in the latter we introduce in an abstract setting the uniqueness condition and we establish some properties that will be used later. Then, in Sections 4 and 5 we prove that the homomorphisms that satisfy the uniqueness condition in the categories Alg_Σ of partial Σ-algebras and $\mathrm{Alg}_{\Sigma,\Upsilon}$ of Υ-total Σ-algebras, respectively, are those closed and injective homomorphisms whose images satisfy the aforementioned mce-property. Finally, Section 6 is devoted to the study of some properties of the mce-property as mentioned above.

This paper is based on the first-named author's PhD Thesis [16], although it contains some results not included therein, and should be considered as a sequel of [17] and [5].

2 Preliminaries

We assume the reader familiar with the basic notions and methods of the theory of partial algebras, as introduced for instance in [3, Chap. I]. Nevertheless, to ease his or her task, we recall in this section some basic definitions and facts about partial algebras, and we take the opportunity to fix some notations and conventions to be used throughout this paper, usually without any further notice.

2.1 Signatures. A *signature* is a triple $\Sigma = (S, \Omega, \eta)$, where S is a non-empty set of *sorts*, Ω is a set of *operation symbols* and $\eta : \Omega \to S^* \times S$ is the *arity mapping*, that sends every $\varphi \in \Omega$ to its *arity* $(\omega(\varphi), \sigma(\varphi)) \in S^* \times S$.

An operation symbol φ is *n-ary* when the length of $\omega(\varphi)$ is n. Let $\Omega^{(n)}$ denote the set of all n-ary operation symbols in Σ and set $\Omega^{(+)} = \Omega - \Omega^{(0)}$.

Let us fix for the rest of this section a signature $\Sigma = (S, \Omega, \eta)$.

2.2 Algebras. A *partial Σ-algebra* is a structure $\mathbf{A} = (A, (\varphi^{\mathbf{A}})_{\varphi\in\Omega})$, where:

- $A = (A_s)_{s\in S}$ is an *S-set* (i.e., an S-indexed family of sets), called the *carrier set* of the algebra; we shall also say that the algebra $\mathbf{A}$ is *supported* on A. For every $s \in S$, the set A_s is called the *carrier set of sort s* of $\mathbf{A}$.
- For every $\varphi \in \Omega$, $\varphi^{\mathbf{A}} : A^{\omega(\varphi)} \to A_{\sigma(\varphi)}$ is a partial mapping, called generically an *operation* in $\mathbf{A}$ (where[2] $A^\lambda = \{\emptyset\}$ and $A^{s_1\cdots s_p} = A_{s_1} \times \cdots \times A_{s_p}$ for every

---

[2] The symbol λ stands for the empty word.

$s_1 \dots s_p \in S^+$). We shall denote by $\operatorname{dom} \varphi^{\mathbf{A}} \subseteq A^{\omega(\varphi)}$ the domain of the partial mapping $\varphi^{\mathbf{A}}$.

An operation $\varphi^{\mathbf{A}}$ in a partial algebra **A** is *total* when it is a total mapping, and *discrete* when it is the empty mapping. When a nullary operation in an algebra **A** is total, we say that it is *defined* in **A**, and we identify it with its image. A partial Σ-algebra is *total* or *discrete* when all operations in it are so.

In the rest of this paper, given a partial algebra denoted by a capital letter in boldface type (**A**, **B**, etc.), we shall always denote, usually without any further notice, its carrier set by the same capital letter, but in slanted type (A, B, etc.); an operation in it by superscripting the operation symbol with the algebra's name ($\varphi^{\mathbf{A}}, \psi^{\mathbf{B}}, \dots$); and, if necessary, its carrier set of a given sort by the same capital letter in slanted type, but with the sort as a subscript (A_s, B_t, etc.). Nevertheless, and in order to lighten the notations, we shall often skip all subscripts corresponding to sorts in the names of the carriers of the algebras, the components of the homomorphisms or the congruences (see below) etc., provided there is no danger of confusion.

2.3 ***Subalgebras.*** Let $\mathbf{A} = (A, (\varphi^{\mathbf{A}})_{\varphi\in\Omega})$ and $\mathbf{B} = (B, (\varphi^{\mathbf{B}})_{\varphi\in\Omega})$ be two partial Σ-algebras with $B \subseteq A$, i.e., with $B_s \subseteq A_s$ for every $s \in S$.

The algebra **B** is a *relative subalgebra* of **A** when it satisfies the following condition: for every $\varphi \in \Omega$ and for every $\underline{b} \in B^{\omega(\varphi)}$, $\underline{b} \in \operatorname{dom} \varphi^{\mathbf{B}}$ if and only if $\underline{b} \in \operatorname{dom} \varphi^{\mathbf{A}} \cap B^{\omega(\varphi)}$ and $\varphi^{\mathbf{A}}(\underline{b}) \in B$, in which case $\varphi^{\mathbf{B}}(\underline{b}) = \varphi^{\mathbf{A}}(\underline{b})$.

The algebra **B** is a *closed subalgebra* of **A** when it is a relative subalgebra and B is a *closed subset* of **A** in the following sense: for every $\varphi \in \Omega$, if $\underline{b} \in \operatorname{dom} \varphi^{\mathbf{A}} \cap B^{\omega(\varphi)}$, then $\varphi^{\mathbf{A}}(\underline{b}) \in B$.

There always exists the least closed subset of a Σ-algebra **A** containing a given subset X of its carrier; we shall denote it by $C_{\mathbf{A}}(X)$ and call it the closed subset of **A** *generated* by X.

2.4 ***Homomorphisms.*** Let $\mathbf{A} = (A, (\varphi^{\mathbf{A}})_{\varphi\in\Omega})$ and $\mathbf{B} = (B, (\varphi^{\mathbf{B}})_{\varphi\in\Omega})$ be two partial Σ-algebras, and let $f : A \to B$ be a *mapping of S-sets*: i.e., an S-indexed family of mappings $f = (f_s)_{s\in S}$ with $f_s : A_s \to B_s$ for every $s \in S$.

The mapping f is a *homomorphism* from **A** to **B** when it preserves the operations in **A**, in the following sense: for every $\varphi \in \Omega$, if $\underline{a} \in \operatorname{dom} \varphi^{\mathbf{A}}$, then $f(\underline{a}) \in \operatorname{dom} \varphi^{\mathbf{B}}$ and $f_{\sigma(\varphi)}(\varphi^{\mathbf{A}}(\underline{a})) = \varphi^{\mathbf{B}}(f(\underline{a}))$ (where $f(\underline{a})$ denotes the image of $\underline{a}$ under the mapping $A^{\omega(\varphi)} \to B^{\omega(\varphi)}$ induced by f).

A homomorphism $f : \mathbf{A} \to \mathbf{B}$ is *closed* with respect to $\varphi \in \Omega$ when, for every $\underline{a} \in A^{\omega(\varphi)}$, if $f(\underline{a}) \in \operatorname{dom} \varphi^{\mathbf{B}}$, then $\underline{a} \in \operatorname{dom} \varphi^{\mathbf{A}}$. A homomorphism is *closed* when it is closed with respect to all operation symbols in the signature. Notice that if $\varphi^{\mathbf{A}}$ is total, then a homomorphism $f : \mathbf{A} \to \mathbf{B}$ is always closed with respect to φ.

If $\mathbf{B}$ is a relative subalgebra of $\mathbf{A}$, the homomorphism $\mathbf{B} \to \mathbf{A}$ given by the set-theoretical inclusion $B \subseteq A$ is called an *embedding*, and it is a closed homomorphism if and only if $\mathbf{B}$ is a closed subalgebra, in which case we call it a *closed embedding*; we shall usually denote an embedding by means of a hooked arrow $\mathbf{A} \hookrightarrow \mathbf{B}$.

Let Alg_Σ (respectively, TAlg_Σ) be the category of all partial (respectively, total) Σ-algebras with homomorphisms as morphisms. In Alg_Σ, the isomorphisms are the closed and bijective homomorphisms, the epimorphisms are those homomorphisms whose set-theoretical image generates the target algebra, and the monomorphisms are the injective homomorphisms (see Props. 2.4.6, 3.6.1 and 2.10.3 in [3], respectively).

2.5 Congruences. Let $\mathbf{A} = (A, (\varphi^{\mathbf{A}})_{\varphi \in \Omega})$ be a partial Σ-algebra and let θ be an *equivalence relation* on its carrier: i.e., an S-indexed family $(\theta_s)_{s \in S}$ of equivalence relations, each θ_s on the corresponding A_s.

The relation θ is a *congruence* on $\mathbf{A}$ when it is compatible with the operations defined in $\mathbf{A}$, in the following sense: for every $\varphi \in \Omega^{(+)}$, say with $\omega(\varphi) = s_1 \ldots s_p \in S^+$, if $\underline{a} = (a_1, \ldots, a_p), \underline{b} = (b_1, \ldots, b_p) \in \operatorname{dom} \varphi^{\mathbf{A}}$ are such that $(a_i, b_i) \in \theta_{s_i}$ for every $i = 1, \ldots, p$, then $(\varphi^{\mathbf{A}}(\underline{a}), \varphi^{\mathbf{A}}(\underline{b})) \in \theta_{\sigma(\varphi)}$.

Given a relation X on the carrier of a partial Σ-algebra $\mathbf{A}$, there always exists the least congruence on $\mathbf{A}$ containing X; we shall denote it by $\theta_{\mathbf{A}}(X)$ and call it the congruence on $\mathbf{A}$ *generated* by X.

Given a congruence θ on a partial Σ-algebra $\mathbf{A} = (A, (\varphi^{\mathbf{A}})_{\varphi \in \Omega})$, the *quotient* algebra

$$\mathbf{A}/\theta = (A/\theta, (\varphi^{\mathbf{A}/\theta})_{\varphi \in \Omega})$$

(where $(A/\theta)_s = A_s/\theta_s$ for every $s \in S$) is defined in the following way: for every $\varphi \in \Omega$,

- if $\varphi \in \Omega^{(0)}$, then $\varphi^{\mathbf{A}/\theta}$ is defined if and only if $\varphi^{\mathbf{A}}$ is defined, and when they are both defined, $\varphi^{\mathbf{A}/\theta} = [\varphi^{\mathbf{A}}]_{\theta_{\sigma(\varphi)}}$ (the equivalence class of $\varphi^{\mathbf{A}}$ modulo $\theta_{\sigma(\varphi)}$);
- if $\omega(\varphi) = s_1 \ldots s_p \in S^+$ and $[\underline{a}] = ([a_1]_{\theta_{s_1}}, \ldots, [a_p]_{\theta_{s_p}}) \in (A/\theta)^{\omega(\varphi)}$, then $[\underline{a}] \in \operatorname{dom} \varphi^{\mathbf{A}/\theta}$ if and only if there exists $\underline{a'} = (a'_1, \ldots, a'_p) \in \operatorname{dom} \varphi^{\mathbf{A}}$ such that $(a_i, a'_i) \in \theta_{s_i}$ for every $i = 1, \ldots, p$; and if this is the case, then $\varphi^{\mathbf{A}/\theta}([\underline{a}]) = [\varphi^{\mathbf{A}}(\underline{a'})]_{\theta_{\sigma(\varphi)}}$.

Given a congruence θ on a partial Σ-algebra $\mathbf{A}$, we denote by $\pi_\theta : \mathbf{A} \to \mathbf{A}/\theta$ the corresponding *quotient homomorphism*, which sends every element $a \in A_s$, $s \in S$, to its equivalence class $[a]_{\theta_s}$.

If $f : \mathbf{A} \to \mathbf{B}$ is a homomorphism, then the relation $\ker f$ defined by $(\ker f)_s = \ker f_s = \{(a, a') \in A_s^2 \mid f_s(a) = f_s(a')\}$, for every $s \in S$, is a congruence on $\mathbf{A}$.

2.6 Pushouts. Let $f : \mathbf{K} \to \mathbf{A}$ and $g : \mathbf{K} \to \mathbf{B}$ be two homomorphisms of partial Σ-algebras. Let $\Sigma^{(+)} = (S, \Omega^{(+)}, \eta\big|_{\Omega^{(+)}})$ be the signature obtained from Σ by removing its nullary operation symbols and let $\mathbf{A}^{(+)}$ and $\mathbf{B}^{(+)}$ be the $\Sigma^{(+)}$-*reducts* of $\mathbf{A}$ and $\mathbf{B}$: the partial $\Sigma^{(+)}$-algebras obtained by simply omitting the nullary operations in them.

The *disjoint sum* of $\mathbf{A}^{(+)}$ and $\mathbf{B}^{(+)}$ is the partial $\Sigma^{(+)}$-algebra

$$\mathbf{A}^{(+)} \oplus \mathbf{B}^{(+)} = (A \sqcup B, (\varphi^{\mathbf{A}^{(+)} \oplus \mathbf{B}^{(+)}})_{\varphi \in \Omega^{(+)}})$$

with carrier set the disjoint union[3] $A \sqcup B$ of the carrier sets A and B of $\mathbf{A}$ and $\mathbf{B}$, and operations defined in the following way: for every $\varphi \in \Omega^{(+)}$,

$$\operatorname{dom} \varphi^{\mathbf{A}^{(+)} \oplus \mathbf{B}^{(+)}} = \operatorname{dom} \varphi^{\mathbf{A}} \sqcup \operatorname{dom} \varphi^{\mathbf{B}}$$

and if $\underline{a} \in \operatorname{dom} \varphi^{\mathbf{A}}$ (respectively, $\underline{b} \in \operatorname{dom} \varphi^{\mathbf{B}}$), then $\varphi^{\mathbf{A}^{(+)} \oplus \mathbf{B}^{(+)}}(\underline{a}) = \varphi^{\mathbf{A}}(\underline{a})$ (respectively, $\varphi^{\mathbf{A}^{(+)} \oplus \mathbf{B}^{(+)}}(\underline{b}) = \varphi^{\mathbf{B}}(\underline{b})$).

Let now $\theta(f, g)$ be the congruence on $\mathbf{A}^{(+)} \oplus \mathbf{B}^{(+)}$ generated by the relation

$$\{(f(x), g(x)) \mid x \in K\} \cup \{(\varphi_0^{\mathbf{A}}, \varphi_0^{\mathbf{B}}) \mid \varphi_0 \in \Omega^{(0)},\ \varphi_0^{\mathbf{A}}, \varphi_0^{\mathbf{B}} \text{ defined}\}.$$

Let $\mathbf{P} = (P, (\varphi^{\mathbf{P}})_{\varphi \in \Omega})$ be the partial Σ-algebra whose $\Sigma^{(+)}$-reduct is the quotient

$$(\mathbf{A}^{(+)} \oplus \mathbf{B}^{(+)})/\theta(f, g)$$

and whose nullary operations are defined as follows: for every $\varphi_0 \in \Omega^{(0)}$, $\varphi_0^{\mathbf{P}}$ is defined if and only if $\varphi_0^{\mathbf{A}}$ or $\varphi_0^{\mathbf{B}}$ is defined, in which case $\varphi_0^{\mathbf{P}}$ is its equivalence class modulo $\theta(f, g)$.

Let finally $\tilde{g} : \mathbf{A} \to \mathbf{P}$ and $\tilde{f} : \mathbf{B} \to \mathbf{P}$ be the homomorphisms given by the restrictions to $\mathbf{A}$ and $\mathbf{B}$ of the quotient mapping $A \sqcup B \to (A \sqcup B)/\theta(f, g)$. Then, the cocone

$$(\mathbf{P}, \tilde{g} : \mathbf{A} \to \mathbf{P}, \tilde{f} : \mathbf{B} \to \mathbf{P})$$

is a pushout of f and g in Alg_Σ.

3 The uniqueness condition

Let us start by formalizing what we understand by the satisfaction of the uniqueness condition in a category $\mathcal{C}$.

---

[3] Formally, we define $A \sqcup B$ as $A \times \{1\} \cup B \times \{2\}$, but in order to simplify the notations we shall identify A and B with their images in $A \sqcup B$.

Definition 1. A morphism $f : K \to A$ in a category $\mathcal{C}$ satisfies the *uniqueness condition* (for pushout complements) *in* $\mathcal{C}$ when, for each morphism $m : A \to D$ in $\mathcal{C}$, if f and m have a pushout complement in $\mathcal{C}$, then it is unique up to an isomorphism over K and D, in the sense that if $(C, g : K \to C, d : C \to D)$ and $(C', g' : K \to C', d' : C' \to D)$ are two pushout complements of f and m, then there exists an isomorphism $h : C' \to C$ such that $h \circ g' = g$ and $d \circ h = d'$.

$$\begin{tikzcd}
& K \arrow[r, "f"] \arrow[d, "g"'] \arrow[ddl, bend right, "g'"'] & A \arrow[d, "m"] \\
& C \arrow[r, "d"'] & D \\
C' \arrow[ur, dotted, "h"] \arrow[urr, bend right, "d'"'] & &
\end{tikzcd}$$

If the left-hand side morphism l of a DPO rule $P = (L \xleftarrow{l} K \xrightarrow{r} R)$ in $\mathcal{C}$ satisfies the uniqueness condition in this category, then every two derived objects by the application of P to a given object G through a given morphism m are isomorphic, because the corresponding context objects are isomorphic over G and K, and then the right-hand side pushout preserves this isomorphism. On the other hand, if l does not satisfy the uniqueness condition, then there are at least two different applications of P to some object G' through some morphism m' such that the corresponding context objects are not isomorphic, something that can result (depending, of course, on the right-hand side morphism r of P) in the existence of two non-isomorphic objects derived from G' by the application of P through m'.

Let us establish now some general results that shall be used in the next sections. In them, and in the sequel, given a fixed class $\mathcal{M}_0$ of morphisms in a category $\mathcal{C}$, by a *pushout* $\mathcal{M}_0$*-complement* of two morphisms $f : K \to A$ and $m : A \to D$ we understand a pushout complement $(B_0, g_0 : K \to B_0, d_0 : B_0 \to D)$ of them with "bottom" morphism d_0 in $\mathcal{M}_0$.

Proposition 2. *Let $\mathcal{C}$ be a category with a factorization system $(\mathcal{E}, \mathcal{M})$ consisting of the class $\mathcal{E}$ of all its epimorphisms and* a *class $\mathcal{M}$ of monomorphisms. Let $\mathcal{M}_0 \subseteq \mathcal{M}$ be a subclass of $\mathcal{M}$ such that, for every $d : A \to B$ in $\mathcal{M}$, there exist an isomorphism $h : A \to A_0$ and a morphism $d_0 : A_0 \to B$ in $\mathcal{M}_0$ such that $d = d_0 \circ h$. Then:*

i) For every pair of morphisms $f : K \to A$ and $m : A \to D$, if they have a pushout complement in $\mathcal{C}$, then they have some pushout $\mathcal{M}_0$-complement.

ii) If a morphism $f : K \to A$ satisfies the uniqueness condition in $\mathcal{C}$, then in *every pushout square in $\mathcal{C}$ with top morphism f*

$$\begin{tikzcd}
K \arrow[r, "f"] \arrow[d, "g"'] & A \arrow[d, "g'"] \\
B \arrow[r, "f'"'] & D
\end{tikzcd}$$

the bottom morphism f' belongs to $\mathcal{M}$.

iii) Assume now that every pair of morphisms in $\mathcal{C}$ having a pushout complement, has exactly one *pushout $\mathcal{M}_0$-complement. Then, a morphism $f : K \to A$ satisfies the uniqueness condition in $\mathcal{C}$ if and only if in every pushout square in $\mathcal{C}$ with top morphism f the bottom morphism belongs to $\mathcal{M}$.*

Proof. i) Given any pushout square in $\mathcal{C}$

$$\begin{tikzcd}
K \arrow[r, "f"] \arrow[d, "g"'] & A \arrow[d, "m"] \\
C \arrow[r, "f'"'] & D
\end{tikzcd}$$

let $f' = d_0 \circ f'_0$ be a factorization of f' with $f'_0 : \mathbf{C} \to \mathbf{B}_0$ in $\mathcal{E}$ and $d_0 : \mathbf{B}_0 \to \mathbf{D}$ in $\mathcal{M}$. Since every morphism in $\mathcal{M}$ is an isomorphism followed by a morphism in $\mathcal{M}_0$, and an epimorphism followed by an isomorphism is still an epimorphism, we can assume that $d_0 \in \mathcal{M}_0$.

In this way, we obtain the following commutative diagram

$$\begin{tikzcd}
K \arrow[r, "f"] \arrow[d, "g"'] \arrow[dr, phantom, "(1)"] & A \arrow[d, "m"] \arrow[dd, bend left=60, "m"] \\
C \arrow[r, "f'"'] \arrow[d, "f'_0"'] \arrow[dr, phantom, "(2)"] & D \arrow[d, "\mathrm{Id}"] \\
B_0 \arrow[r, "d_0"'] & D
\end{tikzcd}$$

where (1) is a pushout square by hypothesis and (2) is a pushout square because f'_0 is an epimorphism. Thus, the outer square (1)+(2) in this diagram is a pushout square, too. Denoting $f'_0 \circ g$ by $g_0 : K \to B_0$, we obtain in this way a pushout $\mathcal{M}_0$-complement $(B_0, g_0 : K \to B_0, d_0 : B_0 \to D)$ of f and m.

ii) Assume that $f : K \to A$ satisfies the uniqueness condition in $\mathcal{C}$ and let

$$\begin{tikzcd}
K \arrow[r, "f"] \arrow[d, "g"'] & A \arrow[d, "g'"] \\
B \arrow[r, "f'"'] & D
\end{tikzcd}$$

be a pushout square in $\mathcal{C}$. Then, $f : K \to A$ and $g' : A \to D$ have a pushout complement in $\mathcal{C}$, and therefore they have a pushout $\mathcal{M}_0$-complement

$$(B_0, g_0 : K \to B_0, d_0 : B_0 \to D).$$

Since f satisfies the uniqueness condition, there exists an isomorphism $h : B \to B_0$ such that $f' = d_0 \circ h$, which implies that $f' \in \mathcal{M}$.

iii) We only have to prove that, under the extra hypothesis, the "if" implication holds. Thus, assume that f is such that in every pushout square in $\mathcal{C}$ whose top morphism is f, the bottom morphism belongs to $\mathcal{M}$. Let $m : A \to D$ be a morphism in $\mathcal{C}$ such that f and m have a pushout complement, and let $(B, g : K \to B, f' : B \to D)$ be any pushout complement of them. By assumption, $f' \in \mathcal{M}$ and therefore $f' = d \circ h$ for some isomorphism $h : B \to C$ and some morphism $d : C \to D$ in $\mathcal{M}_0$.

Reasoning as in the proof of the previous point, we obtain that

$$\begin{tikzcd}
K \arrow[r, "f"] \arrow[d, "h \circ g"'] & A \arrow[d, "m"] \\
C \arrow[r, "d"'] & D
\end{tikzcd}$$

is a pushout square with $d \in \mathcal{M}_0$ and thus, since by assumption the pushout $\mathcal{M}_0$-complement

$$(B_0, g_0 : K \to B_0, d_0 : B_0 \to D)$$

of f and m is unique, we have that $C = B_0$ and hence $d = d_0$ and $h \circ g = g_0$ as well. Thus, we have an isomorphism $h : B \to B_0$ such that $d_0 \circ h = f'$ and $h \circ g = g_0$.

This shows that every pushout complement of f and m is isomorphic over K and D to their pushout $\mathcal{M}_0$-complement, which implies that all pushout complements of f and m are isomorphic over K and D. □

Corollary 3. *Under the general assumptions in Proposition 2, if a morphism $f : K \to A$ in $\mathcal{C}$ satisfies the uniqueness condition, then it must belong to $\mathcal{M}$.*

Proof. Consider the following pushout square in $\mathcal{C}$:

$$\begin{tikzcd}
K \arrow[r, "f"] \arrow[d, "\mathrm{Id}_K"'] & A \arrow[d, "\mathrm{Id}_A"] \\
K \arrow[r, "f"'] & A
\end{tikzcd}$$

If f satisfies the uniqueness condition, then, by point (ii) in the last proposition, it must belong to $\mathcal{M}$, as the bottom homomorphism in this pushout square. □

Corollary 4. *With the notations of Proposition 2, and under the hypothesis in point (iii) in it, if $f : K \to A$ satisfies the uniqueness condition in $\mathcal{C}$ and if*

$$\begin{tikzcd}
K \arrow[r, "f"] \arrow[d, "g"'] & A \arrow[d, "g'"] \\
B \arrow[r, "f'"'] & D
\end{tikzcd}$$

is a pushout square, then f' also satisfies the uniqueness condition in $\mathcal{C}$.

Proof. Assume that f' does not satisfy the uniqueness condition. Then, there exists a pushout square in $\mathcal{C}$

$$\begin{tikzcd}
B \arrow[r, "f'"] \arrow[d, "h"'] & D \arrow[d, "h'"] \\
C \arrow[r, "f''"'] & D'
\end{tikzcd}$$

with $f'' \notin \mathcal{M}$. However, if we compose vertically this pushout square to the one in the statement, we obtain a pushout square

$$\begin{tikzcd}
K \arrow[r, "f"] \arrow[d, "h \circ g"'] & A \arrow[d, "h' \circ g'"] \\
C \arrow[r, "f''"'] & D'
\end{tikzcd}$$

in $\mathcal{C}$ with top morphism f and bottom morphism not in $\mathcal{M}$, which entails that f does not satisfy the uniqueness condition, either. □

It is not difficult to produce *ad hoc* categories where the uniqueness condition is not inherited under pushouts.

4 The uniqueness condition in Alg_Σ

In the sequel, and unless otherwise stated, let $\Sigma = (S, \Omega, \eta)$ be a fixed signature and let Alg_Σ be the category of all partial Σ-algebras with homomorphisms as morphisms.

We know from [3, Prop. 10.2.8] that the pair $(\mathcal{E}, \mathcal{M})$, with $\mathcal{E}$ the class of all epimorphisms and $\mathcal{M}$ the class of all closed monomorphisms, is a factorization system in Alg_Σ, and it is clear that every closed monomorphism f factors into an isomorphism followed by a closed embedding (the embedding of the closed subalgebra supported on the image of f). Therefore, taking as $\mathcal{M}_0$ the class of all closed embeddings, the general hypothesis in Proposition 2 is satisfied.

Following [9], we shall call the pushout $\mathcal{M}_0$-complements in Alg_Σ (with $\mathcal{M}_0$ this class of all closed embeddings) *natural pushout complements.*

On the other hand, in [17, Prop. 9] we proved that the hypothesis in Proposition 2.(iii) is also satisfied with respect to these classes $\mathcal{M}$, $\mathcal{E}$ and $\mathcal{M}_0$: that is, if two homomorphisms of partial Σ-algebras $f : \mathbf{K} \to \mathbf{A}$ and $m : \mathbf{A} \to \mathbf{D}$ have a pushout complement in Alg_Σ, then they have one, and only one, natural pushout complement. Therefore, a direct application of Proposition 2 and Corollary 3 proves the following result.

Corollary 5. *i) A homomorphism of partial* Σ*-algebras* $f : \mathbf{K} \to \mathbf{A}$ *satisfies the uniqueness condition in* Alg_Σ *if and only if in every pushout square in* Alg_Σ *with top homomorphism* f*, the bottom homomorphism is closed and injective.*

ii) Every homomorphism of partial Σ*-algebras satisfying the uniqueness condition in* Alg_Σ *is closed and injective.* □

When the signature Σ is unary, point (ii) in this corollary yields also a sufficient condition: a homomorphism of partial algebras $f : \mathbf{K} \to \mathbf{A}$ over a unary signature Σ satisfies the uniqueness condition in Alg_Σ if and only if it is closed and injective; see [6, Lem. 13 and Prop. 14]. But not every closed and injective homomorphism of partial algebras over an arbitrary signature Σ satisfies the uniqueness condition in the corresponding category Alg_Σ, as the following two examples show: Example 6 corresponds to Example 12 in [17], and we recall it here to ease the task of the reader.

Example 6. Let Σ be a one-sorted signature with a binary operation symbol φ. Let $\mathbf{K}$ and $\mathbf{B}$ be the discrete Σ-algebras with carrier sets $K = \{a_1, a_2, b_1, b_2\}$ and $B = \{a_{1,2}, b_1, b_2\}$, respectively, and let $\mathbf{A}$ be the partial Σ-algebra with carrier set $A = \{a_1, a_2, c, b_1, b_2\}$ and the operation φ defined by $\varphi^{\mathbf{A}}(a_1, c) = b_1$ and $\varphi^{\mathbf{A}}(a_2, c) = b_2$. Let $f : \mathbf{K} \to \mathbf{A}$ be the closed embedding of $\mathbf{K}$ into $\mathbf{A}$ and let $g : \mathbf{K} \to \mathbf{B}$ be the homomorphism given by $g(a_1) = g(a_2) = a_{1,2}$, $g(b_1) = b_1$ and $g(b_2) = b_2$.

A pushout of f and g in Alg_Σ is given by the partial Σ-algebra $\mathbf{D}$ with carrier set $D = \{a_{1,2}, c, b_{1,2}\}$ and the operation φ defined by $\varphi^{\mathbf{D}}(a_{1,2}, c) = b_{1,2}$, together with the homomorphisms $f' : \mathbf{B} \to \mathbf{D}$ and $m : \mathbf{A} \to \mathbf{D}$ given by $f'(a_{1,2}) = a_{1,2}$, $f'(b_1) = f'(b_2) = b_{1,2}$, and $m(a_1) = m(a_2) = a_{1,2}$, $m(c) = c$, $m(b_1) = m(b_2) = b_{1,2}$ respectively. Then, f' is closed but not injective, and therefore f does not satisfy the uniqueness condition.

Example 7. Let Σ be as in the previous example. Let $\mathbf{K}$ and $\mathbf{B}$ be the discrete Σ-algebras with carrier sets $K = \{a_1, a_2, b_1\}$ and $B = \{a_{1,2}, b_1\}$, respectively, and let $\mathbf{A}$ be the partial Σ-algebra with carrier set $A = \{a_1, a_2, c, b_1, b_2, d\}$ and the operation φ defined by $\varphi^{\mathbf{A}}(a_1, c) = b_1$, $\varphi^{\mathbf{A}}(a_2, c) = b_2$ and $\varphi^{\mathbf{A}}(b_1, b_2) = d$. Let $f : \mathbf{K} \to \mathbf{A}$ be the closed embedding of $\mathbf{K}$ into $\mathbf{A}$ and let $g : \mathbf{K} \to \mathbf{B}$ be the homomorphism given by $g(a_1) = g(a_2) = a_{1,2}$ and $g(b_1) = b_1$.

A pushout of f and g in Alg_Σ is given by the partial Σ-algebra $\mathbf{D}$ with carrier set $D = \{a_{1,2}, c, b_{1,2}, d\}$ and the operation φ defined by $\varphi^{\mathbf{D}}(a_{1,2}, c) = b_{1,2}$ and $\varphi^{\mathbf{D}}(b_{1,2}, b_{1,2}) = d$, together with the homomorphisms $f' : \mathbf{B} \to \mathbf{D}$ and $m : \mathbf{A} \to \mathbf{D}$ given by $f'(a_{1,2}) = a_{1,2}$, $f'(b_1) = b_{1,2}$, and $m(a_1) = m(a_2) = a_{1,2}$, $m(c) = c$, $m(b_1) = m(b_2) = b_{1,2}$, $m(d) = d$, respectively. Then, f' is injective but not closed, and therefore f does not satisfy the uniqueness condition.

Thus, in order to obtain the uniqueness condition in Alg_Σ, for an arbitrary signature Σ, some extra condition must be added to closedness and injectivity. As we shall see, this extra condition turns out to be given by the following notion.

Definition 8. A closed subalgebra $\mathbf{C}$ of a partial Σ-algebra $\mathbf{A}$ satisfies the *minimal congruence extension property* (the *mce-property*, for short) when, for every congruence θ on $\mathbf{C}$, if $\overline{\theta}$ is the congruence on $\mathbf{A}$ generated by θ, then:

i) $\overline{\theta} \cap (C \times C) = \theta$;
ii) for every $\varphi \in \Omega$, if $(a_1, \ldots, a_n) \in \mathrm{dom}\, \varphi^{\mathbf{A}}$ and $(a_1, c_1), \ldots, (a_n, c_n) \in \overline{\theta}$ for some $c_1, \ldots, c_n \in C$, then there exists some $(c_1', \ldots, c_n') \in \mathrm{dom}\, \varphi^{\mathbf{C}}$ such that $(a_1, c_1'), \ldots, (a_n, c_n') \in \overline{\theta}$.

A *mce-homomorphism* is a closed and injective homomorphism $f : \mathbf{B} \to \mathbf{A}$ such that the closed subalgebra of $\mathbf{A}$ supported on $f(B)$ satisfies the mce-property. In other words, mce-homomorphisms are those homomorphisms that factor into an isomorphism followed by the closed embedding of a closed subalgebra satisfying the mce-property.

Next lemma shows the real content of the definition of this mce-property.

Lemma 9. *A closed subalgebra* $\mathbf{C}$ *of a partial* Σ*-algebra* $\mathbf{A}$ *satisfies the mce-property if and only if, for every congruence* θ *on* $\mathbf{C}$*, if* $\overline{\theta}$ *is the congruence on* $\mathbf{A}$ *generated by* θ*, then the homomorphism* $\mathbf{C}/\theta \to \mathbf{A}/\overline{\theta}$ *induced by the closed embedding* $\mathbf{C} \hookrightarrow \mathbf{A}$*, defined by* $[c]_\theta \mapsto [c]_{\overline{\theta}}$ *for every* $c \in C$*, is closed and injective.*

Proof. Condition (i) in the definition of the mce-property is equivalent to the injectivity of the homomorphism $\mathbf{C}/\theta \to \mathbf{A}/\overline{\theta}$, while condition (ii) is equivalent to its closedness. $\square$

With the notations of Definition 8, if all operations in Σ are unary, then $\bar{\theta} = \theta \cup \Delta_A$, and therefore in this case every closed subalgebra satisfies the mce-property. This does no longer happen in the arbitrary case. For instance, in Examples 6 and 7, the closed subalgebras $\mathbf{K}$ of the algebras $\mathbf{A}$ do not satisfy the mce-property: in both cases, the least congruence on $\mathbf{A}$ containing $\ker g$ was $\ker m$ (cf. Lemma 10 below), and then in Example 6 condition (i) in Definition 8 is not satisfied (because $(b_1, b_2) \in \ker m \cap (K \times K)$ but $(b_1, b_2) \notin \ker g$) while in Example 7 it is condition (ii) in *loc. cit.* which is not satisfied (because $(b_1, b_1), (b_2, b_1) \in \ker m$, $b_1 \in K$, and $(b_1, b_2) \in \operatorname{dom} \varphi^{\mathbf{A}}$, but $\operatorname{dom} \varphi^{\mathbf{K}} = \emptyset$). These examples also show that conditions (i) and (ii) in the definition of the mce-property are independent of each other.

The connection between pushouts in Alg_Σ and the mce-property stems from the following lemma.

Lemma 10. *Let* $\mathbf{C}$ *be a closed subalgebra of a partial* Σ*-algebra* $\mathbf{A}$*, let* θ *be a congruence on* $\mathbf{C}$ *and* $\overline{\theta}$ *the congruence on* $\mathbf{A}$ *generated by it, and let* $\pi_\theta : \mathbf{C} \to \mathbf{C}/\theta$ *and* $\pi_{\overline{\theta}} : \mathbf{A} \to \mathbf{A}/\overline{\theta}$ *be the corresponding quotient homomorphisms. If* $\widehat{\imath} : \mathbf{C}/\theta \to \mathbf{A}/\overline{\theta}$ *is the homomorphism induced by the closed embedding* $i : \mathbf{C} \hookrightarrow \mathbf{A}$*, then the commutative square*

$$\begin{tikzcd}
\mathbf{C} \arrow[r, hook, "i"] \arrow[d, "\pi_\theta"'] & \mathbf{A} \arrow[d, "\pi_{\overline{\theta}}"] \\
\mathbf{C}/\theta \arrow[r, "\widehat{\imath}"'] & \mathbf{A}/\overline{\theta}
\end{tikzcd}$$

is a pushout square in Alg_Σ.

Proof. Let $p : \mathbf{A} \to \mathbf{D}$ and $q : \mathbf{C}/\theta \to \mathbf{D}$ be two homomorphisms of partial Σ-algebras such that $p \circ i = q \circ \pi_\theta$. If $(c, c') \in \theta$, then $p(c) = q([c]_\theta) = q([c']_\theta) = p(c')$. Therefore, $\theta \subseteq \ker p$ and hence $\overline{\theta} \subseteq \ker p$. Then, by the diagram completion lemma [3, Cor. 2.7.2], there exists a unique homomorphism $\overline{p} : \mathbf{A}/\overline{\theta} \to \mathbf{D}$ such that $\overline{p} \circ \pi_{\overline{\theta}} = p$. And this homomorphism also satisfies that $\overline{p} \circ \widehat{\imath} = q$, because

$$\overline{p} \circ \widehat{\imath}([c]_\theta) = \overline{p} \circ \widehat{\imath} \circ \pi_\theta(c) = \overline{p} \circ \pi_{\overline{\theta}}(c) = p(c) = q([c]_\theta)$$

for every $c \in C$. □

Next proposition shows that bottom homomorphisms of pushout squares with mce top homomorphisms are always closed and injective: this, together with the previous lemma, will be the main ingredient in the proof that the uniqueness condition in Alg_Σ is to be mce.

Proposition 11. *Let* $f : \mathbf{K} \to \mathbf{A}$ *be a mce-homomorphism. If*

$$\begin{tikzcd}
\mathbf{K} \arrow[r, "f"] \arrow[d, "g"'] & \mathbf{A} \arrow[d, "m"] \\
\mathbf{B} \arrow[r, "f'"'] & \mathbf{D}
\end{tikzcd}$$

is a pushout square in Alg_Σ*, then* $f' : \mathbf{B} \to \mathbf{D}$ *is closed and injective.*

Proof. Let $\mathbf{C}$ be the closed subalgebra of $\mathbf{A}$ supported on $C = f(K)$, so that $f : \mathbf{K} \to \mathbf{C}$ is an isomorphism and $\mathbf{C}$ satisfies the mce-property in $\mathbf{A}$. Let $i : \mathbf{C} \hookrightarrow \mathbf{A}$ be the corresponding closed embedding, and let $g' : \mathbf{C} \to \mathbf{B}$ denote the composition $g \circ f^{-1}$. Then, in the following commutative diagram

$$\begin{tikzcd}
\mathbf{K} \arrow[r, "f"'] \arrow[rr, bend left, "f"] \arrow[d, "g"'] \arrow[dr, phantom, "(1)"] & \mathbf{C} \arrow[r, hook, "i"'] \arrow[d, "g'"] \arrow[dr, phantom, "(2)"] & \mathbf{A} \arrow[d, "m"] \\
\mathbf{B} \arrow[r, "\mathrm{Id}_{\mathbf{B}}"] \arrow[rr, bend right, "f'"'] & \mathbf{B} \arrow[r, "f'"] & \mathbf{D}
\end{tikzcd}$$

square (1) is a pushout square because $f : \mathbf{K} \to \mathbf{C}$ is an isomorphism, and the outer square is a pushout square by assumption. Then, by the usual pushout decomposition property, square (2) is a pushout square, too. Thus, to prove that $f' : \mathbf{B} \to \mathbf{D}$ is closed and injective, we only have to prove it in square (2): i.e., it is enough to prove that if $i : \mathbf{C} \hookrightarrow \mathbf{A}$ is the closed embedding of a closed subalgebra satisfying the mce-property and

$$\begin{tikzcd}
\mathbf{C} \arrow[r, hook, "i"] \arrow[d, "g"'] & \mathbf{A} \arrow[d, "m"] \\
\mathbf{B} \arrow[r, "f'"'] & \mathbf{D}
\end{tikzcd}$$

is a pushout square in Alg_Σ, then $f' : \mathbf{B} \to \mathbf{D}$ is closed and injective.

Set $\theta = \ker g$, and consider the factorization of $g : \mathbf{C} \to \mathbf{B}$ as $\overline{g} \circ \pi_\theta$, where $\pi_\theta : \mathbf{C} \to \mathbf{C}/\theta$ denotes the corresponding quotient homomorphism and $\overline{g} : \mathbf{C}/\theta \to \mathbf{B}$ stands for the injective homomorphism defined by $[c]_\theta \mapsto g(c)$ for every $c \in C$. Let $\overline{\theta}$ be the congruence on $\mathbf{A}$ generated by θ and let $\pi_{\overline{\theta}} : \mathbf{A} \to \mathbf{A}/\overline{\theta}$ be the corresponding quotient homomorphism. Consider the following diagram

$$\begin{tikzcd}
\mathbf{C} \arrow[r, hook, "i"] \arrow[d, "\pi_\theta"'] \arrow[dd, bend right=60, "g"'] \arrow[dr, phantom, "(1)"] & \mathbf{A} \arrow[d, "\pi_{\overline{\theta}}"] \arrow[dd, bend left=60, "m"] \\
\mathbf{C}/\theta \arrow[r, "\widehat{\imath}"'] \arrow[d, "\overline{g}"'] \arrow[dr, phantom, "(2)"] & \mathbf{A}/\overline{\theta} \arrow[d, "h"] \\
\mathbf{B} \arrow[r, "f'"'] & \mathbf{D}
\end{tikzcd}$$

where $\widehat{\imath} : \mathbf{C}/\theta \to \mathbf{A}/\overline{\theta}$ is the homomorphism induced by the embedding $i : \mathbf{C} \hookrightarrow \mathbf{A}$, square (1) is a pushout square by Lemma 10, and $h : \mathbf{A}/\overline{\theta} \to \mathbf{D}$ is the unique homomorphism such that $h \circ \pi_{\overline{\theta}} = m$ and $h \circ \widehat{\imath} = f' \circ \overline{g}$, which exists by the universal property of pushouts. Since $\mathbf{C}$ satisfies the mce-property in $\mathbf{A}$, Lemma 9 guarantees that $\widehat{\imath} : \mathbf{C}/\theta \to \mathbf{A}/\overline{\theta}$ is closed and injective.

Then, since the outer square in the previous diagram is also a pushout square, by the pushout decomposition property we have that square (2) is a pushout square in Alg_Σ as well. In this pushout square, $\widehat{\imath}$ is closed and injective and $\overline{g}$ is injective. Then, f' is also closed and injective by [17, Cor. 2]. $\square$

We have finally the following result.

Theorem 12. *A homomorphism of partial Σ-algebras $f : \mathbf{K} \to \mathbf{A}$ satisfies the uniqueness condition in* Alg_Σ *if and only if it is a mce-homomorphism.*

Proof. If f is a mce-homomorphism, then by Proposition 11, every pushout square with top homomorphism f has its bottom homomorphism closed and injective, and therefore f satisfies the uniqueness condition by Corollary 5.

Conversely, let $f : \mathbf{K} \to \mathbf{A}$ be a homomorphism that satisfies the uniqueness condition. Then, by Corollary 5, it must be closed and injective. Assume now that f is closed and injective, but not mce. Then, it is equal to an isomorphism $f_0 : \mathbf{K} \to \mathbf{C}$ followed by a closed embedding $i : \mathbf{C} \hookrightarrow \mathbf{A}$ of a closed subalgebra that does not satisfy the mce-property.

Thus, there exists a congruence θ on $\mathbf{C}$ that does not satisfy conditions (i) or (ii) in Definition 8. As we have seen in Lemma 9, this is equivalent to the fact that the homomorphism $\widehat{\imath} : \mathbf{C}/\theta \to \mathbf{A}/\overline{\theta}$ (where $\overline{\theta}$ denotes the congruence on $\mathbf{A}$ generated by θ) induced by $i : \mathbf{C} \hookrightarrow \mathbf{A}$ is not closed and injective.

Now, in the following commutative diagram

$$\begin{tikzcd}
\mathbf{K} \arrow[r, "f_0"] \arrow[rr, bend left, "f"] \arrow[d, "\pi_\theta \circ f_0"'] \arrow[dr, phantom, "(1)"] & \mathbf{C} \arrow[r, hook, "i"] \arrow[d, "\pi_\theta"] \arrow[dr, phantom, "(2)"] & \mathbf{A} \arrow[d, "\pi_{\overline{\theta}}"] \\
\mathbf{C}/\theta \arrow[r, "\mathrm{Id}_{\mathbf{C}/\theta}"'] \arrow[rr, bend right, "\widehat{\imath}"'] & \mathbf{C}/\theta \arrow[r, "\widehat{\imath}"'] & \mathbf{A}/\overline{\theta}
\end{tikzcd}$$

(1) is a pushout square because $f_0 : \mathbf{K} \to \mathbf{C}$ is an isomorphism, and (2) is a pushout square by Lemma 10. Then, the outer square in this diagram is a pushout square with top homomorphism f and bottom homomorphism not closed and injective. This contradicts the assumption that f satisfies the uniqueness condition. $\square$

Remark 13. As a consequence of the last theorem and Corollary 4, we have that the mce-property for homomorphisms is inherited under pushouts in Alg_Σ, a fact that sharpens Proposition 11.

Let us prove now that the mce-property simplifies the description of pushout complements, when they exist, and the gluing condition.

Proposition 14. *If $f : \mathbf{K} \to \mathbf{A}$ is a mce-homomorphism of partial Σ-algebras and $m : \mathbf{A} \to \mathbf{D}$ is a homomorphism such that f and m have a pushout in* Alg_Σ*, then their natural pushout complement is given by the closed subalgebra of* $\mathbf{D}$ *supported on*

$$(D - m(A)) \cup m(f(K)).$$

Proof. By [17, Th. 10], the natural pushout complement of f and m in Alg_Σ is

$$(\mathbf{B}_0, g_0 : \mathbf{K} \to \mathbf{B}_0, d_0 : \mathbf{B}_0 \hookrightarrow \mathbf{D})$$

where $\mathbf{B}_0$ is the closed subalgebra of $\mathbf{D}$ supported on

$$B_0 = C_{\mathbf{D}}((D - m(A)) \cup m(f(K)))$$

and d_0 is the corresponding closed embedding. Thus, we only have to prove that if $f : \mathbf{K} \to \mathbf{A}$ is a mce-homomorphism, then $(D - m(A)) \cup m(f(K))$ is a closed subset of $\mathbf{D}$. Since this condition does not depend on f but only on its image $f(K)$, we can assume without any loss of generality that the mce-homomorphism $f : \mathbf{K} \to \mathbf{A}$ is actually the embedding of a closed subalgebra satisfying the mce-property: to simplify the notations, we shall omit the symbol f in the sequel.

Set $\theta = \ker g_0$, let $\overline{\theta}$ be the congruence on $\mathbf{A}$ generated by it, and let $\widehat{f} : \mathbf{K}/\theta \to \mathbf{A}/\overline{\theta}$ be the homomorphism induced by the embedding $\mathbf{K} \hookrightarrow \mathbf{A}$. Since $\mathbf{K}$ satisfies the mce-property in $\mathbf{A}$, this homomorphism is closed and injective. Now, arguing as in the proof of Proposition 11, we obtain the following commutative diagram, where $\pi_\theta : \mathbf{K} \to \mathbf{K}/\theta$ and $\pi_{\overline{\theta}} : \mathbf{A} \to \mathbf{A}/\overline{\theta}$ are the corresponding quotient homomorphisms, $\overline{g}_0 : \mathbf{K}/\theta \to \mathbf{B}_0$ is the injective homomorphism defined by $\overline{g}_0([x]_\theta) = g_0(x)$ for every $x \in K$, and all squares in it are pushout squares:

$$\begin{tikzcd}
\mathbf{K} \arrow[r, hook] \arrow[d, "\pi_\theta"'] \arrow[dd, bend right=60, "g_0"'] & \mathbf{A} \arrow[d, "\pi_{\overline{\theta}}"] \arrow[dd, bend left=60, "m"] \\
\mathbf{K}/\theta \arrow[r, "\widehat{f}"'] \arrow[d, "\overline{g}_0"'] & \mathbf{A}/\overline{\theta} \arrow[d, "\overline{m}"] \\
\mathbf{B}_0 \arrow[r, hook, "d_0"'] & \mathbf{D}
\end{tikzcd}$$

Let now $\varphi \in \Omega$ and $\underline{x} \in \operatorname{dom} \varphi^{\mathbf{D}} \cap ((D - m(A)) \cup m(K))^{\omega(\varphi)}$; we have to prove that $\varphi^{\mathbf{D}}(\underline{x}) \in (D - m(A)) \cup m(K)$. So, assume that $\varphi^{\mathbf{D}}(\underline{x}) \notin (D - m(A)) \cup m(K)$. Then there exists some $a \in A - K$ such that $\varphi^{\mathbf{D}}(\underline{x}) = m(a) = \overline{m}([a]_{\overline{\theta}})$. On the other hand, it is clear that $\varphi^{\mathbf{D}}(\underline{x}) \in B_0$, because $\mathbf{B}_0$ is the closed subalgebra of $\mathbf{D}$ generated by $(D - m(A)) \cup m(K)$. Then, since the bottom square in the diagram above is a pushout square and the homomorphisms $\widehat{f}$ and $\overline{g}_0$ in it are closed and injective and injective, respectively, there exists some $k \in K$ such that $[a]_{\overline{\theta}} = \widehat{f}([k]_\theta)$, i.e. $[a]_{\overline{\theta}} = [k]_{\overline{\theta}}$: see the proof of [17, Cor. 2]. But then

$$\varphi^{\mathbf{D}}(\underline{x}) = m(a) = \overline{m}([a]_{\overline{\theta}}) = \overline{m}([k]_{\overline{\theta}}) = m(k) \in m(K),$$

which contradicts the assumption that $\varphi^{\mathbf{D}}(\underline{x}) \notin (D - m(A)) \cup m(K)$. Therefore, $\varphi^{\mathbf{D}}(\underline{x}) \in (D - m(A)) \cup m(K)$, as we wanted to prove. ☐

Remark 15. It is not difficult to produce an example showing that, given two arbitrary homomorphisms of partial Σ-algebras $f : \mathbf{K} \to \mathbf{A}$ and $m : \mathbf{A} \to \mathbf{D}$ having a pushout complement in Alg_Σ, the set $(D - m(A)) \cup m(f(K))$ need not be closed in $\mathbf{D}$ and the relative subalgebra of $\mathbf{D}$ supported on it need not yield a pushout complement of f and m: see, for instance, [16, Ex. 3.11].

Proposition 16. *Let $f : \mathbf{K} \to \mathbf{A}$ be a mce-homomorphism and $m : \mathbf{A} \to \mathbf{D}$ a homomorphism of partial Σ-algebras. Set $B = (D - m(A)) \cup m(f(K))$, let $\mathbf{B}_0$ be the closed subalgebra of $\mathbf{D}$ supported on $B_0 = C_{\mathbf{D}}(B)$, and let $\vartheta(f,m)$ be the congruence on the disjoint sum $\mathbf{A}^{(+)} \oplus \mathbf{B}_0^{(+)}$ of the $\Sigma^{(+)}$-reducts of $\mathbf{A}$ and $\mathbf{B}_0$ generated by the relation*

$$\{(a, m(a)) \mid a \in f(K)\} \cup \{(\varphi_0^{\mathbf{A}}, \varphi_0^{\mathbf{D}}) \mid \varphi_0 \in \Omega^{(0)},\ \varphi_0^{\mathbf{A}}, \varphi_0^{\mathbf{D}} \textit{ defined}\}.$$

Then, f and m have a pushout complement in Alg_Σ *if and only if they satisfy the following conditions:*

i) B is a closed subset of $\mathbf{D}$ (and thus $B = B_0$).
ii) $\ker m \subseteq \vartheta(f,m)$.
iii) For every $\varphi \in \Omega$, if $\underline{d} \in$ $\mathrm{dom}\,\varphi^{\mathbf{D}}$ *and $\underline{d} \notin B^{\omega(\varphi)}$, then $\underline{d} = m(\underline{a})$ for some $\underline{a} \in$* $\mathrm{dom}\,\varphi^{\mathbf{A}}$.

Proof. We already know from [17, Prop. 11] that f and m have a pushout complement in Alg_Σ if and only if they satisfy the following three properties:

GC1) $\ker m \subseteq \vartheta(f,m)$.
GC2) If $m(a) \in B_0$, then $(a, m(a)) \in \vartheta(f,m)$.
GC3) For every $\varphi \in \Omega$, if $\underline{d} \in \mathrm{dom}\,\varphi^{\mathbf{D}}$ and $\underline{d} \notin B_0^{w(\varphi)}$, then $\underline{d} = m(\underline{a})$ for some $\underline{a} \in \mathrm{dom}\,\varphi^{\mathbf{A}}$.

Notice that (GC1) is equal to (ii) and (if (i) holds) (GC3) is equal to (iii).

Now, if f and m have a pushout complement in Alg_Σ, then they satisfy condition (i) by Proposition 14 and hence, since they also satisfy (GC1) and (GC3), they satisfy conditions (i) to (iii). Conversely, if f and m satisfy conditions (i) and (ii), then they satisfy condition (GC2). Indeed, let $m(a) \in B_0$. Since $B_0 = (D - m(A)) \cup m(f(K))$ by condition (i), there exists some $k \in K$ such that $m(a) = m(f(k))$ and then, by condition (ii), $(a, f(k)) \in \ker m \subseteq \vartheta(f,m)$. Since $(f(k), m(f(k))) \in \vartheta(f,m)$ by definition, this finally entails that $(a, m(a)) = (a, m(f(k))) \in \vartheta(f,m)$, as condition (GC2) requires. Therefore, if f and m satisfy conditions (i), (ii) and (iii), then they satisfy conditions (GC1) to (GC3) above. ☐

In the last proposition, the "if" implication is always valid, without any assumption on the homomorphism f (as the proof shows), but in the "only if" implication the hypothesis that $f : \mathbf{K} \to \mathbf{A}$ is a mce-homomorphism is needed to guarantee condition (i): see Remark 15.

As an application, to close this section we model the transitions of a Deterministic Finite Automaton (a DFA, for short) as DPO transformations of partial algebras of a suitable type, in the spirit of Ehrig-Orejas' proposal for the DADT specification [12]. Other applications of the DPO transformation of partial (and partly total) algebras in the specification of complex systems can be found in Chapters 3 and 4 of M. Llabrés' PhD Thesis [16]. Let us mention that a similar specification of a DFA could be carried over using the DPO transformation of attributed graphs [1], but our solution is slightly simpler.

Recall that a *DFA* is a structure $M = (\mathcal{A}, Q, q_0, F, \delta)$, where $\mathcal{A}$ is its *input alphabet*, Q is its finite set of *states*, $q_0 \in Q$ is its *initial state*, $F \subseteq Q$ is its set of *accepting states*, and finally $\delta : Q \times \mathcal{A} \to Q$ is its *transition mapping*. An *instantaneous description* of a DFA is a pair $(q, w) \in Q \times \mathcal{A}^*$: it essentially means that the DFA is in state q and it still has to process the word w. On the set $Q \times \mathcal{A}^*$ of instantaneous descriptions of a DFA M we define the *transition relation* $\vdash$ in the following way: (q, λ) is not related to any instantaneous description through $\vdash$, and if $w = aw'$ with $a \in \mathcal{A}$ and $\delta(q,a) = q'$, then $(q, w) \vdash (q', w')$. Notice that $\vdash$ is actually a partial mapping. A word $w \in \mathcal{A}^*$ is said to be *accepted* by M when $(q_0, w) \vdash^* (q_F, \lambda)$ for some accepting state $q_F \in F$.

We shall model an instantaneous description of a DFA M, together with the specification of M, by means of a partial algebra of a suitable type Σ_{DFA} in such a way that the transitions between instantaneous descriptions will be given by the application of a DPO rule in $\mathrm{Alg}_{\Sigma_{DFA}}$.

The signature $\Sigma_{DFA} = (S_{DFA}, \Omega_{DFA}, \eta_{DFA})$ will be defined as follows: $S_{DFA} =$

$\{input, alphabet, state\}$, $\Omega_{DFA} = \{\ell, \epsilon, \sigma, \delta\}$, and η_{DFA} is given by

$$\eta_{DFA}(\ell) = (input, alphabet),\ \eta_{DFA}(\epsilon) = (input, state),$$
$$\eta_{DFA}(\sigma) = (input, input),\ \eta_{DFA}(\delta) = (state\, alphabet, state).$$

An instantaneous description of a DFA M will be a partial Σ_{DFA}-algebra $\mathbf{A}$ where the operations take the following meanings and have the following properties:

- Its carrier set of sort *alphabet* is the alphabet $\mathcal{A}$ of M, its carrier set of sort *state* is the set of states Q of M, and its carrier set of sort *input* is a non-empty finite set I of (*input*) *indices.*
- The unary operation $\sigma^{\mathbf{A}}$ is injective and it is defined on the whole I but one element, in such a way that it defines a linear order on I: the index $\sigma^{\mathbf{A}}(i)$ is the successor of the index i.
- $\delta^{\mathbf{A}} : Q \times \mathcal{A} \to Q$ is the transition mapping in M. It is, hence, a total operation.
- $\ell^{\mathbf{A}} : I \to \mathcal{A}$ has the same domain as $\sigma^{\mathbf{A}}$. The word formed by its images with the order defined by $\sigma^{\mathbf{A}}$ is the one in the instantaneous description.
- $\epsilon^{\mathbf{A}} : I \to Q$ is only defined on the least element of I, i.e., on the index that is not the successor of any other index. Its image is the state in the instantaneous description.

Consider now the rule P_{trans} in $\mathrm{Alg}_{\Sigma_{DFA}}$ defined in the following way. Its left-hand side algebra $\mathbf{L}$ has two elements x, y of sort *input*, one element a of sort *alphabet* and two elements q, q' of sort *state*, and the operations in it are defined as follow:

$$\sigma^{\mathbf{L}}(x) = y,\ \ell^{\mathbf{L}}(x) = a,\ \epsilon^{\mathbf{L}}(x) = q,\ \delta^{\mathbf{L}}(q, a) = q'.$$

Its interface algebra $\mathbf{K}$ is the closed subalgebra of $\mathbf{L}$ obtained by removing from it the index x, and the homomorphism $l : \mathbf{K} \to \mathbf{L}$ is the corresponding closed embedding. The closed subalgebra $\mathbf{K}$ satisfies the mce-property in $\mathbf{L}$: indeed, the only possible non-trivial congruence on $\mathbf{K}$ identifies q and q', and its extension to $\mathbf{L}$ only identifies q and q', too. Finally, its right-hand side algebra $\mathbf{R}$ is obtained by adding to $\mathbf{K}$ the operation $\epsilon^{\mathbf{R}}(y) = q'$, and the homomorphism $r : \mathbf{K} \to \mathbf{R}$ is the corresponding embedding. Notice that $\mathbf{L}$ and $\mathbf{R}$ correspond to instantaneous descriptions of DFAs in the sense explained above, while $\mathbf{K}$ does not.

Now let $\mathbf{G}$ be a partial Σ_{DFA}-algebra modelling an instantaneous description of a DFA and let $m : \mathbf{L} \to \mathbf{G}$ be a homomorphism of partial Σ_{DFA}-algebras: notice that it will be injective on the carrier sets of sort *alphabet* (because in $\mathbf{L}$ it is a singleton) and *input* (because $\sigma^{\mathbf{G}}$ is injective), and in particular $\mathbf{G}$

must have at least two input indices, and moreover $m(x)$ must be the least element of $\mathbf{G}$, because $\epsilon^{\mathbf{G}}$ must be defined on it. It is straightforward to check thus that such a homomorphism m always satisfies conditions (i), (ii) and (iii) in Proposition 16 with respect to $l : \mathbf{K} \to \mathbf{L}$, and therefore l and m satisfy the gluing condition in $\mathrm{Alg}_{\Sigma_{DFA}}$.

Then, the context algebra $\mathbf{D}$ of the application of P_{trans} to $\mathbf{G}$ through m is obtained by removing from $\mathbf{G}$ the element $m(x)$, and the derived algebra $\mathbf{H}$ is obtained by adding to this context algebra the operation

$$\epsilon^{\mathbf{H}}(\sigma^{\mathbf{G}}(m(x))) = \delta^{\mathbf{G}}(\epsilon^{\mathbf{G}}(m(x)), \ell^{\mathbf{G}}(m(x))).$$

In other words, if $\mathbf{G}$ models an instantaneous description $(q_0, a_0 w')$ of a DFA M and $\delta(q_0, a_0) = q_1$ in M, then $\mathbf{H}$ models the instantaneous description (q_1, w') of the same automaton, i.e., the image of the previous instantaneous description through $\vdash$.

The fact that P_{trans} cannot be applied to a partial Σ_{DFA}-algebra with only one input index corresponds to the fact that $(q, \lambda) \vdash$ nothing. So, starting with an initial instantaneous configuration of a DFA M, successive applications of P_{trans} end up in an instantaneous configuration of the form (q, λ) and stop. Adding suitable nullary operations to our signature Σ_{DFA} to mean the initial state and the accepting states (and making them defined in all three algebras $\mathbf{L}$, $\mathbf{R}$ and $\mathbf{K}$), we could use the corresponding rule P_{trans} to decide whether a word is accepted by M or not.

5 The uniqueness condition in $\mathrm{Alg}_{\Sigma,\Upsilon}$

In the sequel, and unless otherwise stated, let $\Sigma = (S, \Omega, \eta)$ be a fixed signature and Υ a non-empty subset of Ω.

A partial Σ-algebra $\mathbf{A}$ is said to be Υ-*total* when $\varphi^{\mathbf{A}}$ is total for every $\varphi \in \Upsilon$. The full subcategory of Alg_{Σ} with objects all Υ-total Σ-algebras will be denoted by $\mathrm{Alg}_{\Sigma,\Upsilon}$. If Υ is the set of all operation symbols in Σ, then the Υ-total Σ-algebras are exactly the total Σ-algebras and in particular $\mathrm{Alg}_{\Sigma,\Upsilon} = \mathrm{TAlg}_{\Sigma}$.

In [5] we introduced the DPO transformation of Υ-total Σ-algebras in $\mathrm{Alg}_{\Sigma,\Upsilon}$, and we refer the reader to Sect. 3 therein for the basic properties of these algebras. The notion of free Υ-completions plays a key role in the construction of pushouts in $\mathrm{Alg}_{\Sigma,\Upsilon}$, and we shall recall it here. A *free* Υ-*completion* of a partial Σ-algebra $\mathbf{A}$ is an epimorphism $\ell_{\mathbf{A}} : \mathbf{A} \to \overline{\mathbf{A}}$ of partial Σ-algebras, with $\overline{\mathbf{A}}$ Υ-total, that satisfies the following universal property: for every homomorphism $f : \mathbf{A} \to \mathbf{B}$ of partial Σ-algebras with $\mathbf{B}$ Υ-total, there exists one, and only one, homomorphism $\overline{f} : \overline{\mathbf{A}} \to \mathbf{B}$ such that $\overline{f} \circ \ell_{\mathbf{A}} = f$. In [5, Prop. 8] we described a construction of a free Υ-completion of every partial Σ-algebra $\mathbf{A}$, and two free Υ-completions of $\mathbf{A}$ are always isomorphic over $\mathbf{A}$ because of the universal property they satisfy.

As a consequence of this universal property, we also have that if $f : \mathbf{A} \to \mathbf{B}$ is a homomorphism of partial Σ-algebras and $\ell_{\mathbf{A}} : \mathbf{A} \to \overline{\mathbf{A}}$ and $\ell_{\mathbf{B}} : \mathbf{B} \to \overline{\mathbf{B}}$ are free Υ-completions, then there exists one, and only one, homomorphism of Υ-total Σ-algebras $\overline{f} : \overline{\mathbf{A}} \to \overline{\mathbf{B}}$ such that $\overline{f} \circ \ell_{\mathbf{A}} = \ell_{\mathbf{B}} \circ f$: as a matter of fact, free Υ-completions define an epireflection along the inclusion functor $\mathrm{Alg}_{\Sigma,\Upsilon} \hookrightarrow \mathrm{Alg}_{\Sigma}$.

Pushouts in $\mathrm{Alg}_{\Sigma,\Upsilon}$ are constructed through Υ-free completions of pushouts in Alg_{Σ}, as the following two results, which are (essentially) Proposition 19 and Corollary 21 in [5], show.

Proposition 17. *Let* $f : \mathbf{K} \to \mathbf{A}$ *and* $g : \mathbf{K} \to \mathbf{B}$ *be two homomorphisms of partial* Σ*-algebras and let* $(\mathbf{D}, m : \mathbf{A} \to \mathbf{D}, d : \mathbf{B} \to \mathbf{D})$ *a pushout of them in* Alg_{Σ}. *Let* $\ell_{\mathbf{K}} : \mathbf{K} \to \overline{\mathbf{K}}$, $\ell_{\mathbf{A}} : \mathbf{A} \to \overline{\mathbf{A}}$, $\ell_{\mathbf{B}} : \mathbf{B} \to \overline{\mathbf{B}}$, *and* $\ell_{\mathbf{D}} : \mathbf{D} \to \overline{\mathbf{D}}$ *be free* Υ*-completions of* $\mathbf{K}$, $\mathbf{A}$, $\mathbf{B}$ *and* $\mathbf{D}$, *respectively, and let* $\overline{f} : \overline{\mathbf{K}} \to \overline{\mathbf{A}}$, $\overline{g} : \overline{\mathbf{K}} \to \overline{\mathbf{B}}$, $\overline{m} : \overline{\mathbf{A}} \to \overline{\mathbf{D}}$, *and* $\overline{d} : \overline{\mathbf{B}} \to \overline{\mathbf{D}}$ *be respectively the unique homomorphisms extending* f, g, m *and* d *to these free* Υ*-completions.*

$$\begin{tikzcd}
\overline{\mathbf{K}} \arrow[rrr, "\overline{f}"] \arrow[ddd, "\overline{g}"'] & & & \overline{\mathbf{A}} \arrow[ddd, "\overline{m}"] \\
& \mathbf{K} \arrow[ul, "\ell_{\mathbf{K}}"] \arrow[r, "f"] \arrow[d, "g"'] & \mathbf{A} \arrow[ur, "\ell_{\mathbf{A}}"'] \arrow[d, "m"] & \\
& \mathbf{B} \arrow[r, "d"'] \arrow[dl, "\ell_{\mathbf{B}}"'] & \mathbf{D} \arrow[dr, "\ell_{\mathbf{D}}"] & \\
\overline{\mathbf{B}} \arrow[rrr, "\overline{d}"'] & & & \overline{\mathbf{D}}
\end{tikzcd}$$

Then, $(\overline{\mathbf{D}}, \overline{m} : \overline{\mathbf{A}} \to \overline{\mathbf{D}}, \overline{d} : \overline{\mathbf{B}} \to \overline{\mathbf{D}})$ *is a pushout of* $\overline{f}$ *and* $\overline{g}$ *in* $\mathrm{Alg}_{\Sigma,\Upsilon}$. ☐

Corollary 18. *Let* $f : \mathbf{K} \to \mathbf{A}$ *and* $g : \mathbf{K} \to \mathbf{B}$ *be two homomorphisms of* Υ*-total* Σ*-algebras, let* $(\mathbf{D}, g' : \mathbf{A} \to \mathbf{D}, f' : \mathbf{B} \to \mathbf{D})$ *be a pushout of* f *and* g *in* Alg_{Σ}, *and let* $\ell_{\mathbf{D}} : \mathbf{D} \to \overline{\mathbf{D}}$ *be a free* Υ*-completion of* $\mathbf{D}$. *Then* $(\overline{\mathbf{D}}, \ell_{\mathbf{D}} \circ g' : \mathbf{A} \to \overline{\mathbf{D}}, \ell_{\mathbf{D}} \circ f' : \mathbf{B} \to \overline{\mathbf{D}})$ *is a pushout of* f *and* g *in* $\mathrm{Alg}_{\Sigma,\Upsilon}$. ☐

We also have the following result, which will be used in the sequel.

Proposition 19. *Let* $\mathbf{A}$ *be a* Υ*-total* Σ*-algebra,* $\mathbf{B}$ *a partial* Σ*-algebra,* $f : \mathbf{A} \to \mathbf{B}$ *a closed homomorphism of* Σ*-algebras and* $\ell_{\mathbf{B}} : \mathbf{B} \to \overline{\mathbf{B}}$ *a free* Υ*-completion. Then the composition* $\ell_{\mathbf{B}} \circ f : \mathbf{A} \to \overline{\mathbf{B}}$ *is a closed homomorphism.*

Proof. If $\varphi \in \Upsilon$, then $\varphi^{\mathbf{A}}$ is total, and therefore the homomorphism $\ell_{\mathbf{B}} \circ f : \mathbf{A} \to \overline{\mathbf{B}}$ is closed with respect to it. If $\varphi \notin \Upsilon$, then $f : \mathbf{A} \to \mathbf{B}$ and $\ell_{\mathbf{B}} : \mathbf{B} \to \overline{\mathbf{B}}$ are closed with respect to it by assumption and by condition (FAC2) in [5, Prop. 9], respectively, and therefore the composition $\ell_{\mathbf{B}} \circ f$ is closed with respect to it, too. □

Returning to $\mathrm{Alg}_{\Sigma,\Upsilon}$, in this category the pair $(\mathcal{E}, \mathcal{M})$, with $\mathcal{E}$ the class of all epimorphisms and $\mathcal{M}$ the class of all closed monomorphisms, is again a factorization system, because it is so in Alg_{Σ} and every closed subalgebra of a Υ-total Σ-algebra is again Υ-total, and every closed monomorphism f factors into an isomorphism followed by a closed embedding, because it does so in Alg_{Σ}. Therefore, taking again as $\mathcal{M}_0$ the class of all closed embeddings, the general hypothesis in Proposition 2 is satisfied in $\mathrm{Alg}_{\Sigma,\Upsilon}$; we shall keep on calling the pushout $\mathcal{M}_0$-complements, now in $\mathrm{Alg}_{\Sigma,\Upsilon}$, *natural* pushout complements. On the other hand, in [5, Prop. 27] we proved that the hypothesis for Proposition 2.(iii) is also satisfied in $\mathrm{Alg}_{\Sigma,\Upsilon}$ with respect to these classes $\mathcal{M}$, $\mathcal{E}$ and $\mathcal{M}_0$. Therefore, a direct application of Proposition 2 and Corollary 3 proves the following result.

Corollary 20. *i) A homomorphism of* Υ*-total* Σ*-algebras* $f : \mathbf{K} \to \mathbf{A}$ *satisfies the uniqueness condition in* $\mathrm{Alg}_{\Sigma,\Upsilon}$ *if and only if in every pushout square in* $\mathrm{Alg}_{\Sigma,\Upsilon}$ *with top homomorphism* f*, the bottom homomorphism is closed and injective.*

ii) Every homomorphism of Υ*-total* Σ*-algebras satisfying the uniqueness condition in* $\mathrm{Alg}_{\Sigma,\Upsilon}$ *is closed and injective.* □

This corollary, parallel to Corollary 5, hints that the uniqueness condition in $\mathrm{Alg}_{\Sigma,\Upsilon}$ should be the same as in Alg_{Σ}. Theorem 22 below will show that it is indeed so. Previous to it, we prove an inheritance property for the uniqueness condition under free Υ-completions that has an interest in itself and provides one of the implications in that theorem.

Proposition 21. *Let* $f : \mathbf{K} \to \mathbf{A}$ *be a homomorphism of partial* Σ*-algebras, let* $\ell_{\mathbf{K}} : \mathbf{K} \to \overline{\mathbf{K}}$ *and* $\ell_{\mathbf{A}} : \mathbf{A} \to \overline{\mathbf{A}}$ *be free* Υ*-completions of* $\mathbf{K}$ *and* $\mathbf{A}$*, and let* $\overline{f} : \overline{\mathbf{K}} \to \overline{\mathbf{A}}$ *be the unique homomorphism extending* f*. If* $f : \mathbf{K} \to \mathbf{A}$ *satisfies the uniqueness condition in* Alg_{Σ}*, then* $\overline{f} : \overline{\mathbf{K}} \to \overline{\mathbf{A}}$ *satisfies the uniqueness condition in* $\mathrm{Alg}_{\Sigma,\Upsilon}$.

Proof. Let

$$\begin{tikzcd}
\overline{\mathbf{K}} \arrow[r, "\overline{f}"] \arrow[d, "g"'] & \overline{\mathbf{A}} \arrow[d, "m"] \\
\mathbf{B} \arrow[r, "d"'] & \mathbf{D}
\end{tikzcd}$$

be any pushout square in $\mathrm{Alg}_{\Sigma,\Upsilon}$: by Corollary 20, we want to prove that d is closed and injective. By Proposition 17, we can assume that this pushout has been obtained as the outer square in the following commutative diagram

$$\begin{tikzcd}
\overline{\mathbf{K}} \arrow[rrr, "\overline{f}"] \arrow[ddd, "g"'] & & & \overline{\mathbf{A}} \arrow[ddd, "m"] \\
& \mathbf{K} \arrow[ul, "\ell_{\mathbf{K}}"'] \arrow[r, "f"] \arrow[d, "g \circ \ell_{\mathbf{K}}"'] & \mathbf{A} \arrow[ur, "\ell_{\mathbf{A}}"] \arrow[d, "m'"] & \\
& \mathbf{B} \arrow[r, "d'"'] \arrow[dl, "\mathrm{Id}_{\mathbf{B}}"] & \mathbf{D}' \arrow[dr, "\ell_{\mathbf{D}'}"'] & \\
\mathbf{B} \arrow[rrr, "d"'] & & & \mathbf{D}
\end{tikzcd}$$

where the inner square is a pushout square in Alg_{Σ}, $\ell_{\mathbf{D}'} : \mathbf{D}' \to \mathbf{D}$ is a free Υ-completion, $d = \ell_{\mathbf{D}'} \circ d'$ and $m : \overline{\mathbf{A}} \to \mathbf{D}$ is the unique homomorphism extending $m' : \mathbf{A} \to \mathbf{D}'$. Now, by assumption, d' is closed and injective, and hence d is also injective, and closed by Proposition 19. $\square$

The converse implication in this proposition is in general false: for instance, if Σ is a one-sorted signature with a unary operation symbol φ, $\mathbf{K}$ is the discrete Σ-algebra with carrier set $\{a_0\}$ and $\mathbf{A}$ is the partial Σ-algebra with carrier set $\{a_0, a_1\}$ and all operations discrete except φ, which is defined by $\varphi^{\mathbf{A}}(a_0) = a_1$, then the embedding $\mathbf{K} \hookrightarrow \mathbf{A}$ is not closed and therefore it is not a mce-homomorphism. Now take $\Upsilon = \{\varphi\}$, and the free Υ-completions of $\mathbf{K}$ and $\mathbf{A}$ turn out to be the same and the extension to them of the embedding $\mathbf{K} \hookrightarrow \mathbf{A}$ turns out to be the identity (of course, always up to isomorphism), which satisfies the uniqueness condition in $\mathrm{Alg}_{\Sigma,\Upsilon}$ by Corollary 20.(i).

Theorem 22. *A homomorphism $f : \mathbf{K} \to \mathbf{A}$ of Υ-total Σ-algebras satisfies the uniqueness condition in $\mathrm{Alg}_{\Sigma,\Upsilon}$ if and only if it satisfies the uniqueness condition in Alg_{Σ}.*

Proof. If $f : \mathbf{K} \to \mathbf{A}$ satisfies the uniqueness condition in Alg_{Σ}, then it satisfies the uniqueness condition in $\mathrm{Alg}_{\Sigma,\Upsilon}$ by Proposition 21.

Conversely, assume that $f : \mathbf{K} \to \mathbf{A}$ satisfies the uniqueness condition in $\mathrm{Alg}_{\Sigma,\Upsilon}$, and let

$$\begin{tikzcd}
\mathbf{K} \arrow[r, "f"] \arrow[d, "g"'] & \mathbf{A} \arrow[d, "m"] \\
\mathbf{B} \arrow[r, "\hat{f}"'] & \mathbf{D}
\end{tikzcd}$$

be any pushout square in Alg_{Σ}: by Corollary 5, we want to prove that the bottom homomorphism $\hat{f}$ in it is closed and injective.

Let $\mathbf{C}$ be the relative subalgebra of $\mathbf{B}$ supported on $C = g(K)$, let $d : \mathbf{C} \to \mathbf{B}$ denote the corresponding embedding, and let $\overline{g} : \mathbf{K} \to \mathbf{C}$ denote the homomorphism g in the previous square, understood now with target algebra $\mathbf{C}$. Consider the commutative diagram

$$\begin{tikzcd}
\mathbf{K} \arrow[r, "f"] \arrow[d, "\overline{g}"'] \arrow[dd, bend right=60, "g"'] \arrow[dr, phantom, "(1)"] & \mathbf{A} \arrow[d, "g'"] \arrow[dd, bend left=60, "m"] \\
\mathbf{C} \arrow[r, "f'"] \arrow[d, "d"'] \arrow[dr, phantom, "(2)"] & \mathbf{D}' \arrow[d, "h"] \\
\mathbf{B} \arrow[r, "\hat{f}"'] & \mathbf{D}
\end{tikzcd}$$

where (1) is a pushout square in Alg_{Σ} and $h : \mathbf{D}' \to \mathbf{D}$ is the unique homomorphism such that $h \circ g' = m : \mathbf{A} \to \mathbf{D}$ and $h \circ f' = \hat{f} \circ d : \mathbf{C} \to \mathbf{D}$. Then, since the outer square and square (1) in this diagram are pushout squares in Alg_{Σ}, the usual pushout decomposition property implies that square (2) is also a pushout square in Alg_{Σ}. Since $d : \mathbf{C} \to \mathbf{B}$ is injective, if we prove that $f' : \mathbf{C} \to \mathbf{D}'$ is closed and injective, then, by [17, Cor. 2], we shall obtain that $\hat{f} : \mathbf{B} \to \mathbf{D}$ is also closed and injective and we shall be done.

In other words, we can assume without any loss of generality that in the pushout square in Alg_{Σ}

$$\begin{tikzcd}
\mathbf{K} \arrow[r, "f"] \arrow[d, "g"'] & \mathbf{A} \arrow[d, "m"] \\
\mathbf{B} \arrow[r, "\hat{f}"'] & \mathbf{D}
\end{tikzcd}$$

the homomorphism $g : \mathbf{K} \to \mathbf{B}$ is surjective, and therefore $\mathbf{B}$ is also a Υ-total Σ-algebra because the class of all Υ-total Σ-algebras is closed under homomorphic images. In this case, if we let $\ell_{\mathbf{D}} : \mathbf{D} \to \overline{\mathbf{D}}$ be a free Υ-completion of $\mathbf{D}$, then by Corollary 18 we have that

$$\begin{tikzcd}
\mathbf{K} \arrow[r, "f"] \arrow[d, "g"'] & \mathbf{A} \arrow[d, "\ell_{\mathbf{D}} \circ m"] \\
\mathbf{B} \arrow[r, "\ell_{\mathbf{D}} \circ \hat{f}"'] & \overline{\mathbf{D}}
\end{tikzcd}$$

is a pushout square in $\mathrm{Alg}_{\Sigma,\Upsilon}$, and, since f satisfies the uniqueness condition in $\mathrm{Alg}_{\Sigma,\Upsilon}$, the bottom homomorphism $\ell_{\mathbf{D}} \circ \hat{f} : \mathbf{B} \to \overline{\mathbf{D}}$ in this square is closed and injective. This implies that $\hat{f} : \mathbf{B} \to \mathbf{D}$ is also injective, and it is closed by [3, Prop. 2.4.5.(e)]. $\square$

Therefore, as a direct consequence of Proposition 12 and Theorem 22, we have the following result.

Corollary 23. *A homomorphism* $f : \mathbf{K} \to \mathbf{A}$ *of* Υ*-total* Σ*-algebras satisfies the uniqueness condition in* $\mathrm{Alg}_{\Sigma,\Upsilon}$ *if and only if it is a* mce-homomorphism *in the sense of Definition 8: closed, injective, and for every congruence* θ *on the closed subalgebra* $f(\mathbf{K})$ *of* $\mathbf{A}$ *supported on* $f(K)$*, if* $\overline{\theta}$ *is the congruence on* $\mathbf{A}$ *generated by* θ*, then:*

i) $\overline{\theta} \cap (f(K) \times f(K)) = \theta$*;*

ii) For every $\varphi \in \Omega - \Upsilon$*, if* $(a_1, \ldots, a_n) \in \mathrm{dom}\, \varphi^{\mathbf{A}}$ *and* $(a_1, c_1), \ldots, (a_n, c_n) \in \overline{\theta}$ *for some* $c_1, \ldots, c_n \in f(K)$*, then there exists some* $(c'_1, \ldots, c'_n) \in \mathrm{dom}\, \varphi^{f(\mathbf{K})}$ *such that* $(a_1, c'_1), \ldots, (a_n, c'_n) \in \overline{\theta}$*.* □

Notice that in point (ii) in the last statement it was enough to consider only operation symbols not belonging to Υ, because the operations belonging to Υ are total on $f(\mathbf{K})$ (it is a closed subalgebra of a Υ-total Σ-algebra, and therefore Υ-total) and then there is no need to impose conditions on the domains of these operations. In particular, if $\Upsilon = \Omega$, which corresponds to consider arbitrary total algebras, condition (ii) is always satisfied, and therefore the uniqueness condition in TAlg_Σ reduces, up to isomorphism, to the congruence extension property as it is usually defined for total algebras.

Corollary 24. *A homomorphism* $f : \mathbf{K} \to \mathbf{A}$ *of total* Σ*-algebras satisfies the uniqueness condition in* TAlg_Σ *if and only if it is closed, injective, and the closed subalgebra* $f(\mathbf{K})$ *of* $\mathbf{A}$ *supported on* $f(K)$ *satisfies the* congruence extension property*: for every congruence* θ *on* $f(\mathbf{K})$*, if* $\overline{\theta}$ *is the congruence on* $\mathbf{A}$ *generated by* θ*, then* $\overline{\theta} \cap (f(K) \times f(K)) = \theta$*.* □

This kind of condition on closed subalgebras has been often studied in the literature on total algebras, but mainly from the point of view of finding sufficient (usually, equational) conditions on a total algebra $\mathbf{A}$ to guarantee that all its closed subalgebras satisfy this congruence extension property. For instance, all closed subalgebras of abelian groups and distributive lattices satisfy the congruence extension property, while not all closed subalgebras of semigroups, groups or lattices satisfy it. Thus, an embedding of abelian groups always satisfies the uniqueness condition in the category of total algebras of the corresponding type, but an embedding of groups need not satisfy it.

Notice that Corollary 4 and Theorem 22 entail that the mce-property for homomorphisms is also inherited under pushouts in $\mathrm{Alg}_{\Sigma,\Upsilon}$. And also notice that Theorem 22, together with [6, Lem. 13 and Prop. 14], imply that if Σ is a unary signature, then, for every subset of operation symbols Υ, a homomorphism of Υ-total Σ-algebras satisfies the uniqueness property in $\mathrm{Alg}_{\Sigma,\Upsilon}$ if and only if it is closed and injective: for total algebras, this result is well known.

To close this section, we want to mention that if $f : \mathbf{K} \to \mathbf{A}$ is a mce-homomorphism of Υ-total Σ-algebras and $m : \mathbf{A} \to \mathbf{D}$ is any homomorphism of Υ-total Σ-algebras such that m and f have a pushout complement in $\mathrm{Alg}_{\Sigma,\Upsilon}$, then (as we already saw in Alg_Σ) the set $(D - m(A)) \cup m(f(K))$ is closed in $\mathbf{D}$, but (against what happens in Alg_Σ) the closed subalgebra of $\mathbf{D}$ supported on this set need not yield a pushout complement of f and m in $\mathrm{Alg}_{\Sigma,\Upsilon}$: see [16, Cor. 4.59, Exs. 4.61 and 4.62]. Actually, the fact that f satisfies the mce property does not simplify at all the description of pushout complements of f and a composable homomorphism m in $\mathrm{Alg}_{\Sigma,\Upsilon}$ for an arbitrary signature Σ, which is much more involved than in Alg_Σ or for unary signatures: see [5].

6 On the satisfaction of the mce-property

In this section we consider the problem of the satisfaction of the mce-property. In §6.1, we prove that if a closed subalgebra of a partial algebra satisfies the mce-property for all finitely generated congruences, then it satisfies it for all congruences, and we show that, against what happens for total algebras [2], in general it is not enough to satisfy the mce-property for principal congruences to guarantee its satisfaction for all congruences. Then, in §6.2 we introduce a simple algebraic condition on a closed subalgebra that implies the satisfaction of the mce-property. Unfortunately, this condition is not necessary and, actually, we don't know a simple, algebraic characterization of those closed subalgebras satisfying the mce-property, even for total algebras.

6.1 The mce-property for finitely-generated congruences

A congruence on a partial algebra $\mathbf{A}$ is *finitely generated* (respectively, *principal*) when it is the congruence on $\mathbf{A}$ generated by a finite relation (respectively, by a single ordered pair).

Proposition 25. *Let* $\mathbf{A}$ *be a partial* Σ*-algebra. A closed subalgebra* $\mathbf{B}$ *of* $\mathbf{A}$ *satisfies the mce-property if and only if conditions (i) and (ii) in Definition 8 are satisfied for every finitely generated congruence* θ *on* $\mathbf{B}$.

Proof. We only have to prove that if conditions (i) and (ii) in Definition 8 are satisfied for every finitely generated congruence on $\mathbf{B}$, then they are also satisfied for every congruence on $\mathbf{B}$.

To prove condition (i), let θ be any congruence on $\mathbf{B}$ and let $\overline{\theta}$ be the congruence on $\mathbf{A}$ generated by θ. By [3, Prop. 2.5.4], $\overline{\theta}$ is equal to the union of the congruences on $\mathbf{A}$ generated by finite subsets of θ:

$$\overline{\theta} = \bigcup \{\theta_{\mathbf{A}}(X) \mid X \text{ a finite subset of } \theta\}.$$

And by assumption we have that $\theta_{\mathbf{A}}(X) \cap (B \times B) = \theta_{\mathbf{B}}(X)$ for every finite subset X of θ, because $\theta_{\mathbf{A}}(X) = \theta_{\mathbf{A}}(\theta_{\mathbf{B}}(X))$.

Thus, if $(b_1, b_2) \in \overline{\theta} \cap (B \times B)$, then there exists some finite subset $X \subseteq \theta$ such that $(b_1, b_2) \in \theta_{\mathbf{A}}(X) \cap (B \times B)$ and then $(b_1, b_2) \in \theta_{\mathbf{B}}(X) \subseteq \theta$. Since, on the other hand, $\theta \subseteq \overline{\theta}$, this entails the equality $\overline{\theta} \cap (B \times B) = \theta$ required by condition (i).

As far as condition (ii) goes, let $\varphi \in \Omega$, $(a_1, \ldots, a_n) \in \operatorname{dom} \varphi^{\mathbf{A}}$ and $(b_1, \ldots, b_n) \in B^{\omega(\varphi)}$ be such that $(a_i, b_i) \in \overline{\theta}$ for every $i = 1, \ldots, n$. Then, there exist finite subsets $X_1, \ldots, X_n$ of θ such that $(a_i, b_i) \in \theta_{\mathbf{A}}(X_i)$ for every $i = 1, \ldots n$. Taking $X = \bigcup_{i=1}^{n} X_i$, we have that $(a_i, b_i) \in \theta_{\mathbf{A}}(X)$ for every $i = 1, \ldots, n$.

But then $\theta_{\mathbf{A}}(X)$ is equal to the congruence on $\mathbf{A}$ generated by $\theta_{\mathbf{B}}(X)$ and thus, since by assumption condition (ii) is satisfied for finitely generated congruences on $\mathbf{B}$, there exist $b'_1, \ldots, b'_n \in B$ such that $(b'_1, \ldots, b'_n) \in \operatorname{dom} \varphi^{\mathbf{B}}$ and $(a_1, b'_1), \ldots, (a_n, b'_n) \in \theta_{\mathbf{A}}(X) \subseteq \overline{\theta}$, as required by condition (ii). ☐

Reasoning as in [2], it can be proved that if condition (i) in Definition 8 is satisfied for every principal congruence on a closed subalgebra $\mathbf{B}$ of a partial algebra $\mathbf{A}$, then it is satisfied for every congruence on $\mathbf{B}$. But it is not true for condition (ii), and thus we cannot reduce the mce-property to principal congruences, as the following example shows.

Example 26. Let Σ be a one-sorted signature with only one operation symbol φ, which is binary. Let $\mathbf{A}$ be the Σ-algebra with carrier set

$$A = \{a_1, a_2, a_3, c_1, c_2, b_1, b_2, b_3, b_4, d\}$$

and operation φ defined by $\varphi^{\mathbf{A}}(a_1, c_1) = b_1$, $\varphi^{\mathbf{A}}(a_2, c_1) = b_2$, $\varphi^{\mathbf{A}}(a_2, c_2) = b_3$, $\varphi^{\mathbf{A}}(a_3, c_2) = b_4$ and $\varphi^{\mathbf{A}}(b_2, b_3) = d$, and let $\mathbf{B}$ be the closed subalgebra of $\mathbf{A}$ supported on $B = \{a_1, a_2, a_3, b_1, b_4\}$, which is discrete.

It turns out that $\mathbf{B}$ satisfies the mce-property for all principal congruences. Indeed, if θ is a principal congruence on $\mathbf{B}$ generated by any ordered pair different from (a_1, a_2) or (a_2, a_3) (or their inverse pairs), then $\overline{\theta} = \theta \cup \Delta_A$ and the mce-property is clearly satisfied in this case. If $\theta = \theta_{\mathbf{B}}((a_1, a_2))$, then

$$\overline{\theta} = \theta \cup \{(b_1, b_2), (b_2, b_1)\} \cup \Delta_A,$$

while if $\theta = \theta_{\mathbf{B}}((a_2, a_3))$, then

$$\overline{\theta} = \theta \cup \{(b_3, b_4), (b_4, b_3)\} \cup \Delta_A,$$

and it is straightforward to check that the mce-property is also satisfied in these cases.

However, if we consider the congruence θ on $\mathbf{B}$ generated by $\{(a_1,a_2),(a_2,a_3)\}$, then we have that $(b_1,b_2),(b_4,b_3) \in \overline{\theta}$, and $(b_2,b_3) \in \operatorname{dom}\varphi^{\mathbf{A}}$ and $b_1,b_4 \in B$, but there is no $(b,b') \in \operatorname{dom}\varphi^{\mathbf{B}}$ such that $(b,b_2),(b',b_3) \in \overline{\theta}$.

6.2 Strongly convex subalgebras

Given a partial Σ-algebra $\mathbf{A}$ with carrier set $A=(A_s)_{s\in S}$, the *algebraic quasi-order* on $\mathbf{A}$ is the reflexive and transitive closure $\preceq_{\mathbf{A}}^*$ on $\bigcup_{s\in S} A_s$ of the following relation (see [3, §5.5]):

> $a' \prec_{\mathbf{A}} a$ if and only if $a=\varphi^{\mathbf{A}}(a_1,\ldots,a',\ldots,a_m)$ for some operation $\varphi\in\Omega$ and some tuple $(a_1,\ldots,a',\ldots,a_m)\in\operatorname{dom}\varphi^{\mathbf{A}}$.

In general, $\preceq_{\mathbf{A}}^*$ need not be a partial order.

Definition 27. A subset X of the carrier set of a partial Σ-algebra $\mathbf{A}$ is *strongly convex* when, for every operation symbol φ and for every $(a_1,\ldots,a_n) \in \operatorname{dom}\varphi^{\mathbf{A}}$, if $\varphi^{\mathbf{A}}(a_1,\ldots,a_n)\in X$ and there exists some $x\in X$ such that $x \preceq_{\mathbf{A}}^* a_{i_0}$ for some $i_0=1,\ldots,n$, then $a_1,\ldots,a_n\in X$.

It is obvious that every initial segment of a partial algebra is a strongly convex subset: if X is an initial segment, then $\varphi^{\mathbf{A}}(a_1,\ldots,a_n)\in X$ implies $a_1,\ldots,a_n\in X$ without any other assumption. On the other hand, every strongly convex subset X is clearly *convex* in the sense of [18]: if $\varphi^{\mathbf{A}}(a_1,\ldots,a_n) \in X$ and there exist some $x\in X$ such that $x\preceq_{\mathbf{A}}^* a_{i_0}$ for some $i_0=1,\ldots,n$, then $a_{i_0}\in X$.

These implications are strict: not every strongly convex subset is an initial segment and, if the signature contains some operation symbol that is at least binary, not every convex subset is strongly convex, as the following simple example shows.

Example 28. Let Σ be a one-sorted signature with a binary operation φ. Let $\mathbf{A}$ be the partial Σ-algebra with carrier set $A=\{a,c,b\}$ and the operation φ defined by $\varphi^{\mathbf{A}}(a,c)=b$. Then $K=\{b\}$ is a strongly convex closed subset of A that is not an initial segment, while $K'=\{a,b\}$ is a convex closed subset of A that is not strongly convex.

In the next proposition we show that the strong convexity implies the mce-property. Unfortunately, the converse implication does not hold: if in Example 7 we denote by $\mathbf{K}'$ the discrete Σ-algebra with carrier set $K'=\{a_1,a_2,d\}$, then it is a closed subalgebra of $\mathbf{A}$ that satisfies the mce-property, but K' is not a strongly convex subset of $\mathbf{A}$. Moreover, convexity and the mce-property are not related to each other: for instance, this subset K' is not convex in $\mathbf{A}$

either, while in Example 6 the set K is a convex closed subset of $\mathbf{A}$, but the corresponding closed subalgebra $\mathbf{K}$ does not satisfy the mce-property in $\mathbf{A}$.

Proposition 29. *If* $\mathbf{K} \hookrightarrow \mathbf{A}$ *is a closed subalgebra supported on a strongly convex set, then* $\mathbf{K}$ *satisfies the mce-property in* $\mathbf{A}$.

Proof. Let K be a strongly convex closed subset of a partial Σ-algebra $\mathbf{A}$, let θ be a congruence relation on $\mathbf{K}$ and let $\overline{\theta}$ be the congruence on $\mathbf{A}$ generated by θ. We must prove that θ satisfies conditions (i) and (ii) in Definition 8.

To do that, we shall prove first, using algebraic induction, that $\overline{\theta} \cap (K \times A) = \theta$, i.e., that, for every $(a, b) \in \overline{\theta}$,

(*) if $a \in K$ or $b \in K$, then $(a, b) \in \theta$.

Indeed:

- If $(a, b) \in \theta$ or $a = b$, then (a, b) clearly satisfies (*).
- If (a, b) satisfies (*), so does (b, a).
- If $(a, c), (c, b) \in \overline{\theta}$ satisfy (*) and, say, $a \in K$, then $(a, c) \in \theta$ and hence $c \in K$, which implies that $(c, b) \in \theta$ as well, and thus, by transitivity, that $(a, b) \in \theta$.
- Assume that there exist $\varphi \in \Omega$ and $(x_1, \ldots, x_n), (y_1, \ldots, y_n) \in \operatorname{dom} \varphi^{\mathbf{A}}$ such that $(x_1, y_1), \ldots, (x_n, y_n) \in \overline{\theta}$ satisfy (*), and let $a = \varphi^{\mathbf{A}}(x_1, \ldots, x_n)$ and $b = \varphi^{\mathbf{A}}(y_1, \ldots, y_n)$. Let $U_{\mathbf{A}}(K)$ be the least final segment of $\mathbf{A}$ (with respect to the algebraic quasi-ordering) containing X:

  $$U_{\mathbf{A}}(K) = \{a \in A \mid k \preceq^*_{\mathbf{A}} a \text{ for some } k \in K\}.$$

  Since $U_{\mathbf{A}}(K)$ is a final segment of $\mathbf{A}$, $(U_{\mathbf{A}}(K))^2 \cup \Delta_A$ is a congruence on $\mathbf{A}$, and it contains θ because it contains $K \times K$: therefore,

  $$\overline{\theta} \subseteq (U_{\mathbf{A}}(K))^2 \cup \Delta_A.$$

  In particular, for every $i = 1, \ldots, n$, we have that either $x_i = y_i$ or $x_i, y_i \in U_{\mathbf{A}}(K)$. If $x_i = y_i$ for every i, then $a = b$. Assume now that $x_{i_0} \neq y_{i_0}$, for some i_0, and that $a \in K$. Then, since K is strongly convex in $\mathbf{A}$ and $x_{i_0} \in U_{\mathbf{A}}(K)$, it turns out that $x_1, \ldots, x_n \in K$ and hence, by the algebraic induction hypothesis, $(x_i, y_i) \in \theta$ for every $i = 1, \ldots, n$. Finally, since $\mathbf{K}$ is a closed subalgebra of $\mathbf{A}$ and θ is a congruence relation on $\mathbf{K}$, this implies that $(a, b) \in \theta$.

Now it is clear that θ satisfies condition (i) in Definition 8. As far as condition (ii) goes, let $\varphi \in \Omega$ and $(a_1, \ldots, a_n) \in \operatorname{dom} \varphi^{\mathbf{A}}$ be such that there exist $c_1, \ldots, c_n \in K$ such that $(a_1, c_1), \ldots, (a_n, c_n) \in \overline{\theta}$. Then, by the property on $\overline{\theta}$ we have just proved, $(a_1, c_1), \ldots, (a_n, c_n) \in \theta$ and in particular $(a_1, \ldots, a_n) \in \operatorname{dom} \varphi^{\mathbf{K}}$. □

References

[1] M. Berthold, I. Fischer, M. Koch, Attributed Graph Transformation with Partial Attribution and its Application to Neural Networks, in: Proc. APPLIGRAPH/GETGRATS Workshop on Graph Transformation Systems (Berlin, 2000), 171–178.

[2] W. J. Blok, D. Pigozzi. On the congruence extension property, Algebra Universalis 38 (1997), 391–394.

[3] P. Burmeister, A Model Theoretic Oriented Approach to Partial Algebras, Mathematical Research 32, Akademie-Verlag 1986.

[4] P. Burmeister, M. Llabrés, F. Rosselló, Pushout Complements for Arbitrary Total Algebras (Extended Abstract), in: Proc. APPLIGRAPH/GETGRATS Workshop on Graph Transformation Systems (Berlin, 2000), 163–170.

[5] P. Burmeister, M. Llabrés, F. Rosselló, Pushout complements for partly total algebras, Mathematical Structures in Computer Science 12 (2002), 177–201.

[6] P. Burmeister, F. Rosselló, J. Torrens, G. Valiente, Algebraic transformation of unary partial algebras I: Double-pushout approach, Theoretical Computer Science 184 (1997), 145–193.

[7] A. Corradini, U. Montanari, F. Rossi, H. Ehrig, R. Heckel, M. Löwe, Algebraic approaches to graph transformation I: Basic concepts and double pushout approach, in: Handbook of Graph Grammars and Computing by Graph Transformation, Vol. I: Foundations, World Scientific, 1997, 163–245.

[8] A. Day, A note on the congruence extension property, Algebra Universalis 1 (1970), 234–235.

[9] H. Ehrig, Introduction to the algebraic theory of graph grammars, in: Proceedings 1st Workshop on Graph Grammars and their Application to Computer Science and Biology, Lecture Notes in Computer Science 73, Springer-Verlag, 1979, 1–69.

[10] H. Ehrig, A. Hable, H.-J. Kreowski, F. Parisi-Presicce, Parallelism and concurrency in high-level replacement systems, Mathematical Structures in Computer Science 1 (1991), 361–404.

[11] H. Ehrig, M. Löwe, Categorical principles, techniques and results for high-level replacement systems in computer science, Applied Categorical Structures 1 (1993), 21–50.

[12] H. Ehrig, F. Orejas, Dynamic Abstract Data Types: An informal proposal, Bull. EATCS 53 (1994), 162–169.

[13] H. Ehrig, M. Pfender, H. J. Schneider, Graph grammars: an algebraic approach, in: 14th Annual IEEE Symposium on Switching and Automata Theory (1973), 167–180.

[14] G. Grätzer, Universal Algebra (Second Edition), Springer Verlag 1979.

[15] Y. Kawahara, Pushout-complements and basic concepts of grammars in toposes, Theoretical Computer Science 77 (1990), 267–289.

[16] M. Llabrés, Double Pushout Transformation of Algebras, PhD Thesis, University of the Balearic Islands, 2001.

[17] M. Llabrés, F. Rosselló, Pushout complements for arbitrary partial algebras, in: Proceedings 6th International Workshop on Theory and Application of Graph Transformation TAGT'98, Lecture Notes in Computer Science 1764, Springer-Verlag, 2000, 131–144.

[18] B. Staruch, Algebraic characterization of classes of partial algebras definable by strong equations and mixed strong and existential equations, Preprint, 1999.

[19] X. Tang, Semigroups with the congruence extension property, Semigroup Forum 56 (1998), 228–264.

[20] X. Tang, On the congruence extension property for semigroups: preservation under homomorphic images, J. Algebra 238 (2001), 411–425.

[21] W. Taylor, Appendix 4: Equational Logic, in: [14].