# Scalar and fuzzy cardinalities of crisp and fuzzy multisets*

Jaume Casasnovas, Francesc Rosselló

Department of Mathematics and Computer Science,
University of the Balearic Islands,
07122 Palma de Mallorca (Spain)
*E-mail:* {`jaume.casasnovas,cesc.rossello`}`@uib.es`

**Abstract**

We define in an axiomatic way scalar and fuzzy cardinalities of finite multisets over $]0,1]$, and we obtain explicit descriptions for them. We show that, for multisets over $]0,1]$ associated to finite fuzzy sets, the cardinalities defined in this paper are equivalent to the cardinalities of the corresponding fuzzy sets previously introduced in the literature. Finally, we also define in an axiomatic way scalar and fuzzy cardinalities of finite fuzzy multisets over any set $X$, and we use the descriptions of the cardinalities of finite multisets over $]0,1]$ to obtain explicit characterizations of the former.

**Keywords:** Multisets, fuzzy multisets, fuzzy bags, generalized natural numbers, cardinality

## 1 Introduction

A (crisp) *multiset*, or *bag*, over a set of *types* $V$ is simply a mapping $d : V \to \mathbb{N}$. The usual interpretation of a multiset $d : V \to \mathbb{N}$ is that it describes a set consisting of $d(v)$ copies of each type $v \in V$, or, more in general, that, if $V$ stands for a family of pairwise disjoint crisp properties, this multiset describes a collection of objects containing, for every property $v$, $d(v)$ members with this property.

A first generalization of multisets under the latter interpretation would be to understand the properties in $V$ as non-crisp, and more specifically as taking values in the unit interval $[0,1]$, but still pairwise disjoint in the sense that if an object satisfies a property $v$ with a certain degree $t > 0$, it cannot satisfy any other property in $V$ with any degree $t' > 0$. Then, one would look for a mathematical object describing a set by means of the specification, for every $v \in V$ and $t \in [0,1]$, of the number of elements in the set satisfying $v$ with degree $t$. This leads in a natural way to the notion of

*This work has been partially supported by the Spanish DGES, project BFM2003-00771.

*fuzzy multiset*, or *fuzzy bag*, over a set $V$, whose members we shall still call *types*, as a mapping $F : V \times [0,1] \to \mathbb{N}$; under the interpretation just discussed, such a fuzzy multiset describes a set consisting of, for each $v \in V$ and $t \in [0,1]$, $F(v,t)$ objects satisfying property $v$ with degree $t$. Of course, other semantics can be attached to the mathematical notion of fuzzy multiset. For instance, in Yager's original interpretation [25], $F : V \times [0,1] \to \mathbb{N}$ describes a set containing, for each $v \in V$ and $t \in [0,1]$, $F(v,t)$ copies of the type $v$ that belong to the set with membership degree $t$.

A fuzzy multiset is *finite* when it takes non-zero values only on a finite subset of $V \times [0,1]$; this would correspond, under the interpretations discussed above, to the finiteness of the set described by the fuzzy multiset. As a matter of fact, as it will be discussed in the next section, we shall actually define a *finite fuzzy multiset* as a mapping $F : V \times ]0,1] \to \mathbb{N}$ —or, equivalently, as a mapping $F : V \to \mathbb{N}^{]0,1]}$— that takes a non-zero value on a finite subset of $V \times [0,1]$, but for the introductory purposes of this section it is not necessary to modify the original definition.

As a generalization of the corresponding concept for crisp multisets and fuzzy sets, cardinalities of fuzzy multisets aim at 'measuring the size' of a fuzzy multiset, and they have found applications for instance in flexible querying of databases [12, 19]. Our specific interest in measuring finite fuzzy multisets stems from their application in the development of a fuzzy version of membrane computing that handles inexact, erroneous copies of the objects used in computations. Without entering into any detail (the interested reader can look up the textbook [17]), membrane systems manipulate finite multisets, and the result of a computation is the cardinal of a finite multiset. Then, fuzzy membrane systems should manipulate finite fuzzy multisets as defined above, and then the result of a computation should be obtained by 'counting' a finite fuzzy multiset. For instance, we have proposed a fuzzy approach to membrane computation with fuzzy multisets where the output fuzzy multiset was measured in an *ad hoc* way [9].

The problem of 'counting' fuzzy sets has generated a lot of literature since Zadeh's first definition of a cardinality of fuzzy sets [26]. In particular, the scalar cardinalities of fuzzy sets, which associate to each finite fuzzy set a positive real number, have been studied from the axiomatic point of view [7, 8, 11, 24] with the aim of capturing different ways of taking into account additive aspects of fuzzy sets like the cardinals of supports, of levels, of cores, etc. In a similar way, the fuzzy cardinalities of fuzzy sets [15, 18, 21, 22, 23], which associate to each finite fuzzy set a convex fuzzy natural number, have also been studied from the axiomatic point of view [6, 10]. Mainly, the axiomatic definition of cardinalities has included on the one hand the consistency with the crisp case and on the other hand the additivity for the additive join of fuzzy sets, as it is found also in the crisp case. The families of fuzzy cardinalities defined axiomatically include Zadeh's decreasing cardinality *FGCount* [26], also used by other authors with alternative names [15], and the increasing cardinality *FLCount*, as well as several modifications of the latter by means of suitable mappings [22].

As far as cardinalities of multisets go, an extension to fuzzy multisets of Zadeh's original definition of the scalar cardinality of fuzzy sets has already been introduced [1, 3, 25]. Furthermore, an extension to multisets of the cardinality *FGCount* for fuzzy sets has been used [4, 5, 19], as well as nonconvex cardinalities of fuzzy multisets [13, 14].

The aim of this paper is the axiomatic definition of scalar and fuzzy cardinalities of fuzzy multisets and a meaningful description of the families of cardinalities obtained in this way. In both cases, the additivity for the sum of multisets and the consistency with the properties of cardinalities in the crisp case has been our main concern. In the scalar case, the family of cardinalities we obtain includes the usual scalar cardinality used in [1, 3, 24], whereas in the fuzzy case it includes the decreasing cardinality $|\ |$ defined in a nonaxiomatic way in [19]; we would like to point out that in [19] the cardinal of the sum of two fuzzy multisets is defined through the extension principle, while in this paper we prove the additivity property for this cardinality.

## 2 Preliminaries

Throughout this paper, the operations $\vee$ and $\wedge$ on the unit interval $[0,1]$ stand respectively for the usual *maximum* and *minimum* operations. Consequently, for every $Y \subseteq [0,1]$, $\bigvee Y$ and $\bigwedge Y$ denote the *supremum* and the *infimum* of $Y$, respectively. Similarly, by the operations $\vee$ and $\wedge$ on the set $\mathbb{N}$ of natural numbers we mean the usual *maximum* and *minimum* operations of natural numbers, respectively.

Given a mapping $f : A \to B$ between two partially ordered sets, we shall say that $f$ is *increasing* when $a_1 \leqslant a_2$ implies $f(a_1) \leqslant f(a_2)$, and that it is *decreasing* when $a_1 \leqslant a_2$ implies $f(a_1) \geqslant f(a_2)$.

Let $X$ be a crisp set. A (*crisp*) *multiset* over $X$ is a mapping $M : X \to \mathbb{N}$, where $\mathbb{N}$ stands for the set of natural numbers including the 0. A good survey of the mathematics of multisets, including their axiomatic foundation, can be found in [2].

A multiset $M$ over $X$ is *finite* if its *support*

$$Supp(M) = \{x \in X | M(x) > 0\}$$

is a finite subset of $X$. We shall denote the sets of all finite multisets and of all finite multisets over a set $X$ by $MS(X)$ and $FMS(X)$, respectively, and by $\perp$ the *null multiset*, defined by $\perp(x) = 0$ for every $x \in X$.

A *singleton* is a multiset over a set $X$ that sends some element $x \in X$ to $1 \in \mathbb{N}$ and all other elements of $X$ to $0 \in \mathbb{N}$; we shall denote such a singleton by $1/x$. More generally, for every $x \in X$ and $n \in \mathbb{N}$, we shall denote by $n/x$ the multiset on $X$ that sends $x$ to $n$ and all other elements of $X$ to 0: in particular, $0/x = \perp$ for every $x \in X$.

For every $A, B \in MS(X)$, their *join* $A \vee B$ and *meet* $A \wedge B$ are, respectively, the multisets over $X$ defined pointwise by

$$(A \vee B)(x) = A(x) \vee B(x) \text{ and } (A \wedge B)(x) = A(x) \wedge B(x), \quad x \in X.$$

The *sum* $A + B$ of two multisets $A, B$ over $X$ is the multiset defined by

$$(A + B)(x) = A(x) + B(x), \quad \text{for every } x \in X.$$

It has been argued [20] that this sum $+$, also called *additive union*, is the right notion of union of multisets. Under the interpretation of multisets as sets of copies of types explained above, this sum corresponds to the disjoint union of sets, as it interprets that all copies of each $x$ in the set represented by $A$ are different from all copies of it in the set represented by $B$. This additive sum has properties quite different from the ordinary union of sets. For instance, the collection of submultisets of a given multiset is not closed under this operation and consequently no sensible notion of complement within this collection exists.

The partial order $\leqslant$ on $MS(X)$ is defined by

$$A \leqslant B \text{ if and only if } A(x) \leqslant B(x) \text{ for every } x \in X.$$

If $A, B \in MS(X)$ are such that $A \leqslant B$, then their *difference* $B - A$ is the multiset defined pointwise by

$$(B - A)(x) = B(x) - A(x) \text{ for every } x \in X.$$

With this definition we have that $A + (B - A) = B$. When $A \nleqslant B$, then it is usual to define $B - A$ by means of

$$(B - A)(x) = (B(x) - A(x)) \vee 0 \text{ for every } x \in X,$$

but in this case the equality $A + (B - A) = B$ does no longer hold.

If $A$ and $B$ are finite, then $A + B$, $A \vee B$, $A \wedge B$, and $B - A$ are also finite.

As we have mentioned in the introduction, a fuzzy multiset is a mapping $F : V \times [0, 1] \to \mathbb{N}$. In our semantics, $V$ stands for a set of disjoint properties and then $F$ describes a set consisting, for each $v \in V$ and $t \in [0, 1]$, of $F(v, t)$ objects that satisfy $v$ with degree $t$. Such a fuzzy multiset is *finite* when it takes a non-zero value only on a finite number of pairs $(v, t) \in V \times [0, 1]$.

In the sequel, we shall assume that the set described by a fuzzy multiset does not contain any element that does not satisfy some $v \in V$ with some non-negative degree. This assumption, together with the assumption that the properties in $V$ are pairwise disjoint, entail that, if the fuzzy multiset $F$ is finite, then, for every $v_0 \in V$, the value $F(v_0, 0)$ must be equal to $\sum_{w \in V - \{v_0\}} \sum_{t > 0} F(w, t)$, because, under these conditions, the equality

$$\sum_{v \in V} \sum_{t > 0} F(v, t) = F(v_0, 0) + \sum_{t > 0} F(v_0, t)$$

holds: both terms in this equality are equal to the number of elements of the set described by $F$. In particular, the restriction of $F$ to $V \times \{0\}$ is determined by the restriction of $F$ to $V\times]0,1]$. Then, since not having to care about the images under finite fuzzy multisets of the elements of the form $(v,0)$ greatly simplifies some of the definitions and results that will be introduced in the main body of this paper, we do just this, and we define a *finite fuzzy multiset* over a set $V$ as a mapping $\overline{M}: V\times]0,1]\to\mathbb{N}$ that takes non-zero value only on a finite number of pairs $(v,t)$. Equivalently, and using the natural bijection $\mathbb{N}^{V\times]0,1]}\cong(\mathbb{N}^{]0,1]})^V$, we can define a *finite fuzzy multiset* over a set $V$ as a mapping $\overline{M}: V\to FMS(]0,1])$ whose *support*

$$Supp(\overline{M})=\{x\in X\mid \overline{M}(x)\neq\perp\}$$

is a finite subset of $X$.

We shall denote the set of all finite fuzzy multisets by $\mathcal{FFMS}(X)$, and by $\overline{\perp}$ the *null* fuzzy multiset defined by $\overline{\perp}(x)=\perp$ for every $x\in X$.

Given two finite fuzzy multisets $\overline{A},\overline{B}$ over $X$, their *sum* $\overline{A}+\overline{B}$, their *join* $\overline{A}\vee\overline{B}$ and their *meet* $\overline{A}\wedge\overline{B}$ are respectively the finite fuzzy multisets over $X$ defined pointwise by

$$\begin{array}{l}(\overline{A}+\overline{B})(x)=\overline{A}(x)+\overline{B}(x)\\(\overline{A}\vee\overline{B})(x)=\overline{A}(x)\vee\overline{B}(x)\\(\overline{A}\wedge\overline{B})(x)=\overline{A}(x)\wedge\overline{B}(x)\end{array}$$

where now the sum, join and meet on the right-hand side of these equalities are operations between multisets; so, for instance, $\overline{A}+\overline{B}: X\to MS(]0,1])$ is the finite fuzzy multiset such that, for every $x\in X$,

$$\begin{array}{rrcl}(\overline{A}+\overline{B})(x): & ]0,1] & \to & \mathbb{N}\\ & t & \mapsto & \overline{A}(x)(t)+\overline{B}(x)(t)\end{array}$$

The partial order $\leqslant$ on $\mathcal{FMS}(X)$ is defined by

$$\overline{A}\leqslant\overline{B}\text{ if and only if }\overline{A}(x)\leqslant\overline{B}(x)\text{ for every }x\in X$$

where the symbol $\leqslant$ in the right-hand side of this equivalence stands for the partial order between crisp multisets defined above.

If $\overline{A},\overline{B}\in\mathcal{FMS}(X)$ are such that $\overline{A}\leqslant\overline{B}$, then their *difference* $\overline{B}-\overline{A}$ is the finite fuzzy multiset defined pointwise by

$$(\overline{B}-\overline{A})(x)=\overline{B}(x)-\overline{A}(x),$$

where, again, the difference in the right hand side term in this equality stands for the difference of crisp multisets defined above. Notice that, if $\overline{A}\leqslant\overline{B}$, then $\overline{A}+(\overline{B}-\overline{A})=\overline{B}$.

When $\overline{A}\nleqslant\overline{B}$, then Rocacher [19] replaces the difference $\overline{B}-\overline{A}$ by the *optimistic difference*

$$\overline{B})-(\overline{A}=\bigvee\{\overline{S}\in\mathcal{FMS}(X)\mid(\overline{A}\wedge\overline{B})+\overline{S}\leqslant\overline{B}\};$$

it is not difficult to check that this optimistic difference is given by

$$\Big(\overline{B}) - (\overline{A}\Big)(x): \begin{array}{ccc} ]0,1] & \to & \mathbb{N} \\ t & \mapsto & (\overline{B}(x)(t) - \overline{A}(x)(t)) \vee 0 \end{array}$$

In this case, it need not be true that the sum of $\overline{A}$ and $\overline{B}) - (\overline{A}$ yields $\overline{B}$.

A *generalized natural number* [23] is a fuzzy subset $\nu : \mathbb{N} \to [0,1]$ of $\mathbb{N}$. To add generalized natural numbers, we shall use the *extended sum* $\oplus$, defined as follows (see for instance [22]): for every $\mu, \nu \in [0,1]^{\mathbb{N}}$,

$$(\nu \oplus \mu)(k) = \bigvee \{\nu(i) \wedge \mu(j) \mid i + j = k\} \text{ for every } k \in \mathbb{N}.$$

It is well known that this extended sum of generalized natural numbers is associative, commutative and that if $\overline{0}$ denotes the generalized natural number that sends 0 to 1 and every $n > 0$ to 0, then $\nu \oplus \overline{0} = \nu$ for every generalized natural number $\nu$. As a consequence of these properties, the extended sum of $m$ generalized natural numbers is well defined:

$$(\nu_1 \oplus \cdots \oplus \nu_m)(i) = \bigvee \{\nu_1(i_1) \wedge \cdots \wedge \nu_m(i_m) \mid i_1 + i_2 + \cdots + i_m = i\}. \qquad (1)$$

Moreover, the extended sum of two increasing (resp., decreasing) generalized natural numbers is again increasing (resp., decreasing).

A generalized natural $\nu$ number is *convex* when $\nu(k) \geqslant \nu(i) \wedge \nu(j)$ for every $i \leqslant k \leqslant j$. Every increasing or decreasing generalized natural number is convex, and the extended sum of two convex generalized natural number is again convex. For these and other properties of generalized natural numbers, see [22].

## 3 Scalar cardinalities of finite multisets over $]0,1]$

We introduce and discuss in this section the notion of scalar cardinality of finite crisp multisets on $]0,1]$. From now on, $\mathbb{R}^+$ stands for the set of all real numbers greater or equal than 0.

**Definition 1.** *A* scalar cardinality *on* $FMS(]0,1])$ *is a mapping* $Sc : FMS(]0,1]) \to \mathbb{R}^+$ *that satisfies the following conditions:*

*(i)* $Sc(A + B) = Sc(A) + Sc(B)$ *for every* $A, B \in FMS(]0,1])$.

*(ii)* $Sc(1/1) = 1$.

**Example 1.** The usual cardinality of finite multisets $|\ | : FMS(]0,1]) \to \mathbb{R}^+$ defined by $|A| = \sum_{t \in Supp(A)} A(t)$ for every $A \in FMS(]0,1])$, is a scalar cardinality.

**Remark 1.** If $Sc : FMS(]0,1]) \to \mathbb{R}^+$ is a scalar cardinality, then $Sc(\perp) = 0$, because

$$1 = Sc(1/1) = Sc((1/1) + \perp) = Sc(1/1) + Sc(\perp) = 1 + Sc(\perp),$$

and if $A \leqslant B$, then $Sc(A) \leqslant Sc(B)$, because in this case

$$Sc(B) = Sc(A + (B - A)) = Sc(A) + Sc(B - A) \geqslant Sc(A).$$

**Remark 2.** We have that if $Sc$ is a scalar cardinality on $FMS(]0,1)]$, then, for every $A, B \in FMS(]0,1])$,

$$Sc(A \vee B) + Sc(A \wedge B) = Sc(A) + Sc(B),$$

because

$$A \vee B + A \wedge B = A + B$$

and then the additivity of scalar cardinalities (condition (i) in Definition 1) applies. In particular, if $A \wedge B = \perp$, then $Sc(A \vee B) = Sc(A) + Sc(B)$.

Next proposition provides a description of all scalar cardinalities on $FMS(]0,1)$.

**Proposition 1.** *A mapping $Sc : FMS(]0,1]) \to \mathbb{R}^+$ is a scalar cardinality if and only if there exists some mapping $f : ]0,1] \to \mathbb{R}^+$ with $f(1) = 1$, such that*

$$Sc(A) = \sum_{t \in Supp(A)} f(t)A(t) \qquad \textit{for every } A \in FMS(]0,1]).$$

*Proof.* Let $Sc$ be a scalar cardinality on $FMS(]0,1])$, and consider the mapping

$$\begin{array}{rccc} f: & ]0,1] & \to & \mathbb{R}^+ \\ & t & \mapsto & Sc(1/t) \end{array}$$

We have that $f(1) = Sc(1/1) = 1$, by condition (ii) in Definition 1. And since every $A \in FMS(]0,1])$ can be decomposed into a sum of singletons, namely,

$$A = \sum_{t \in Supp(A)} \overbrace{1/t + \cdots + 1/t}^{A(t)},$$

condition (i) in Definition 1 implies that

$$Sc(A) = \sum_{t \in Supp(A)} \overbrace{Sc(1/t) + \cdots + Sc(1/t)}^{A(t)} = \sum_{t \in Supp(A)} A(t)f(t).$$

Conversely, let $f\ :]0,1] \to \mathbb{R}^+$ be a mapping such that $f(1) = 1$, and let $Sc_f$ : $FMS(]0,1]) \to \mathbb{R}^+$ be the mapping defined by

$$Sc_f(A) = \sum_{t \in Supp(A)} f(t)A(t)$$

for every $A \in FMS(]0,1])$. Then, this mapping satisfies the defining conditions of scalar cardinalities. Indeed, $Sc_f(1/1) = f(1)\cdot(1/1)(1) = 1$, which proves condition (ii) in Definition 1. As far as condition (i) goes,

$$\begin{aligned} Sc_f(A+B) &= \textstyle\sum_{t\in Supp(A+B)} f(t)(A(t)+B(t)) \\ &= \textstyle\sum_{t\in Supp(A+B)} f(t)A(t) + \sum_{t\in Supp(A+B)} f(t)B(t) \\ &= \textstyle\sum_{t\in Supp(A)} f(t)A(t) + \sum_{t\in Supp(B)} f(t)B(t) = Sc_f(A) + Sc_f(B). \end{aligned}$$

□

From now on, and as we did in the last proof, whenever we want to stress the mapping $f\ :]0,1] \to \mathbb{R}^+$ that *generates* a given scalar cardinality, we shall denote the latter by $Sc_f$. In particular, the scalar cardinality $|\ |$ of Example 1 is the scalar cardinality $Sc_1$ associated to the constant mapping 1; we shall use henceforth this last expression $Sc_1$ to denote it.

Let $Sc_f$ be any scalar cardinality on $FMS(]0,1])$. As we saw in Remark 1, for every $A, B \in FMS(]0,1])$, if $A \leqslant B$, then $Sc_f(A) \leqslant Sc_f(B)$. The converse implication is, of course, false. Let, for instance, $f$ be the constant mapping 1, and let $A$ be the singleton $1/t_0$ and $B$ the singleton $1/t_1$ with $t_0 \neq t_1$. Then $Sc_1(A) = 1 = Sc_1(B)$ but neither $A \leqslant B$ nor $B \leqslant A$.

It is more interesting to point out that, for certain mappings $f$, it may happen that $A \leqslant B$ and $Sc_f(A) = Sc_f(B)$ but $A \neq B$. For instance, let $f\ :]0,1] \to \mathbb{R}^+$ be any mapping such that $f(1) = 1$ and $f(t_0) = 0$ for some $t_0 \neq 1$. Let $A$ be the singleton $1/t_0$ and $B$ the multiset $2/t_0$. Then $A \leqslant B$ and $Sc_f(A) = 0 = Sc_f(B)$, but $A \neq B$.

Actually, sending some element of $]0,1]$ to 0 is unavoidable in order to obtain such a counterexample: the reader can easily prove that if $f\ :]0,1] \to \mathbb{R}^+$ is such that $f(t) > 0$ for every $t \in ]0,1]$, then, for every $A, B \in FMS(]0,1])$, if $A \leqslant B$ and $Sc_f(A) = Sc_f(B)$, then $A = B$.

## 4 Fuzzy cardinalities of finite multisets over $]0,1]$

In this section, we introduce and discuss the notion of fuzzy cardinality of crisp finite multisets over $]0,1]$. Let $\overline{\mathbb{N}}$ denote from now on the set of all convex generalized natural numbers.

**Definition 2.** *A* fuzzy cardinality *on $FMS(]0,1])$ is a mapping $\mathcal{C} : FMS(]0,1]) \to \overline{\mathbb{N}}$ that satisfies the following conditions:*

(i) *(Additivity) For every* $A, B \in FMS(]0,1])$, $\mathcal{C}(A+B) = \mathcal{C}(A) \oplus \mathcal{C}(B)$.

(ii) *(Variability) For every* $A, B \in FMS(]0,1])$ *and for every* $i, j \in \mathbb{N}$, *if* $i > Sc_1(A)$ *and* $j > Sc_1(B)$, *then* $\mathcal{C}(A)(i) = \mathcal{C}(B)(j)$.

(iii) *(Consistency)* $\mathcal{C}(1/1)$ *takes its values in* $\{0,1\}$, *and* $\mathcal{C}(1/1)(1) = 1$.

(iv) *(Monotonicity) If* $t, t' \in ]0,1]$ *are such that* $t \leqslant t'$, *then*

$$\mathcal{C}(1/t)(0) \geqslant \mathcal{C}(1/t')(0) \quad \text{and} \quad \mathcal{C}(1/t)(1) \leqslant \mathcal{C}(1/t')(1).$$

Let us explain our motivation for introducing each one of these axioms. The *additivity property* generalizes the usual additivity of cardinals of sets. We actually consider this property the most characteristic of cardinals. With respect to the *variability property*, we believe that the generalized natural number defined by the fuzzy cardinality of a finite multiset $A$ over $]0,1]$ must take only a finite set of values, and therefore it must be constant from a certain natural number on: it is natural to take the number $Sc_1(A) = \sum_{t \in ]0,1]} A(t)$ as the last place where $\mathcal{C}(A)$ can vary. And then, by analogy with the fuzzy sets case (see Section 5 and [10]) and the particular fuzzy cardinals of fuzzy multisets already introduced in the literature (see, for instance, [19]), we require this constant value $\mathcal{C}(A)(i)$, for $i \geqslant Sc_1(A) + 1$, to be the same for every multiset $A$. The *consistency property* imposes that the cardinality of the singleton $1/1$ represents the crisp natural number 1, in a sense that will be made precise by Corollary 6. Finally, and as far as the *monotonicity property* goes, we impose it to capture the fact that, if $t \leqslant t'$, then $\mathcal{C}(t/1)$ must be 'smaller or equal than' $\mathcal{C}(t'/1)$, for every sensible ordering of convex generalized natural numbers. Indeed, it seems natural to ask to such a sensible ordering on $\overline{\mathbb{N}}$ that if $\mu$ is smaller than $\nu$, then, informally, the increasing branch of $\mu$ lies to the left —or simply above— the increasing branch of $\nu$, and the decreasing branch of $\mu$ lies again to the left —or, in this case, below— the decreasing branch of $\nu$; cf. [22]. Now, as it turns out, this, together with the rest of axioms, would imply that $\mathcal{C}(1/t)(0) \geqslant \mathcal{C}(1/t')(0)$ and $\mathcal{C}(1/t)(1) \leqslant \mathcal{C}(1/t')(1)$, as we require.

The fuzzy cardinality defined in the next example will play a key role henceforth, and, as we shall see in Section 5, it generalizes in a very precise way the usual bracket notation for fuzzy sets.

**Example 2.** Let us consider the function

$$\begin{array}{rrcl} [\,]: & FMS(]0,1]) & \to & [0,1]^{\mathbb{N}} \\ & A & \mapsto & [A] \end{array}$$

where, for every $A \in FMS(]0,1])$,

$$\begin{array}{rrcl} [A]: & \mathbb{N} & \to & [0,1] \\ & i & \mapsto & [A]_i \end{array}$$

is defined by

$$[A]_i = \bigvee \{t \in [0,1] \mid \sum_{t' \geqslant t} A(t') \geqslant i\}.$$

For instance, if $M : ]0,1] \to \mathbb{N}$ is the multiset defined by $M(1/3) = 1$, $M(2/3) = 2$, $M(3/4) = 1$ and $M(t) = 0$ otherwise, then

$$\sum_{t' \geqslant t} M(t') = \begin{cases} 4 & \text{if } t \leqslant \frac{1}{3} \\ 3 & \text{if } \frac{1}{3} < t \leqslant \frac{2}{3} \\ 1 & \text{if } \frac{2}{3} < t \leqslant \frac{3}{4} \\ 0 & \text{if } \frac{3}{4} < t \end{cases}$$

and thus

$$[M]_0 = 1,\ [M]_1 = \tfrac{3}{4},\ [M]_2 = [M]_3 = \tfrac{2}{3},\ [M]_4 = \tfrac{1}{3}$$
$$[M]_i = 0 \text{ for every } i > 4 = Sc_1(A).$$

In general, $[A]$ is decreasing, for every $A \in FMS(]0,1])$: for every $i \leqslant j$,

$$\{t \in [0,1] \mid \sum_{t' \geqslant t} A(t') \geqslant j\} \subseteq \{t \in [0,1] \mid \sum_{t' \geqslant t} A(t') \geqslant i\}$$

and hence

$$[A]_j = \bigvee \{t \in [0,1] \mid \sum_{t' \geqslant t} A(t') \geqslant j\} \leqslant \bigvee \{t \in [0,1] \mid \sum_{t' \geqslant t} A(t') \geqslant i\} = [A]_i.$$

Therefore, $[A] \in \overline{\mathbb{N}}$ for every $A \in FMS(]0,1])$.

The mapping $[\ ]$ satisfies the variability (if $i > Sc_1(A)$, then $[A]_i = \bigvee \emptyset = 0$), the consistency ($[1/1]_0 = [1/1]_1 = 1$ and $[1/1]_i = 0$ for every $i \geqslant 2$) and the monotonicity (for every $t \in ]0,1]$, $[1/t]_0 = 1$ and $[1/t]_1 = t$) conditions. As far as the additivity condition goes, we have that, for every $A, B \in FMS(]0,1])$ and for every $i \in \mathbb{N}$,

$$\begin{aligned}
[A+B]_i &= \bigvee\{t \in ]0,1] \mid \textstyle\sum_{t' \geqslant t}(A+B)(t') \geqslant i\} \\
&= \bigvee\{t \in ]0,1] \mid \textstyle\sum_{t' \geqslant t} A(t') + \sum_{t' \geqslant t} B(t') \geqslant i\} \\
&= \bigvee\{t \in ]0,1] \mid \text{there exist } j,k \in \mathbb{N} \text{ such that } j+k=i \\
&\qquad\qquad \text{and } \textstyle\sum_{t' \geqslant t} A(t') \geqslant j \text{ and } \sum_{t' \geqslant t} B(t') \geqslant k\} \\
&= \bigvee\Big\{\bigvee\{t \in ]0,1] \mid \textstyle\sum_{t' \geqslant t} A(t') \geqslant j \text{ and } \sum_{t' \geqslant t} B(t') \geqslant k\} \\
&\qquad\qquad \Big| \; j,k \in \mathbb{N}, j+k=i\Big\} \\
&= \bigvee\Big\{\bigvee\{t \in ]0,1] \mid \textstyle\sum_{t' \geqslant t} A(t') \geqslant j\} \wedge \bigvee\{t \in ]0,1] \mid \sum_{t' \geqslant t} B(t') \geqslant k\} \\
&\qquad\qquad \Big| \; j,k \in \mathbb{N},\ j+k=i\Big\} \\
&= \bigvee\{[A]_j \wedge [B]_k \mid j,k \in \mathbb{N},\ j+k=i\} = ([A] \oplus [B])(i).
\end{aligned}$$

Therefore, $[\ ]$ is a fuzzy cardinality on $FMS(]0,1])$.

The bracket fuzzy cardinality will lie at the basis of any other fuzzy cardinality on $FMS(]0,1])$, and therefore it will be often useful to have a detailed description of it.

**Lemma 2.** *Let $A : ]0,1] \to \mathbb{N}$ be a finite multiset. If $A = \bot$, then $[A]_0 = 1$ and $[A]_i = 0$ for every $i > 0$. If $Supp(A) = \{t_1, \ldots, t_n\} \neq \emptyset$, with $t_1 < \cdots < t_n$, then, for every $i \geqslant 0$,*

$$[A]_i = \begin{cases} 1 & \textit{if } i = 0 \\ t_n & \textit{if } 0 < i \leqslant A(t_n) \\ t_{n-1} & \textit{if } A(t_n) < i \leqslant A(t_n) + A(t_{n-1}) \\ \vdots & \\ t_s & \textit{if } \sum_{j=s+1}^{n} A(t_j) < i \leqslant \sum_{j=s}^{n} A(t_j) \\ \vdots & \\ t_1 & \textit{if } \sum_{j=2}^{n} A(t_j) < i \leqslant \sum_{j=1}^{n} A(t_j) \\ 0 & \textit{if } \sum_{j=1}^{n} A(t_j) < i \end{cases}$$

*Proof.* The case when $A = \bot$ is obvious, since $\sum_{t' \geqslant t} A(t) = 0$ for every $t \in ]0,1]$. As far as the case when $A \neq \bot$ goes, if $Supp(A) = \{t_1, \ldots, t_n\}$, with $t_1 < \cdots < t_n$, it is straightforward to check that

$$\sum_{t'\geqslant t} A(t') = \begin{cases} \sum_{j=1}^{n} A(t_j) & \text{if } t \in [0, t_1] \\ \sum_{j=2}^{n} A(t_j) & \text{if } t \in ]t_1, t_2] \\ \vdots & \\ \sum_{j=s}^{n} A(t_j) & \text{if } t \in ]t_{s-1}, t_s] \\ \vdots & \\ A(t_n) & \text{if } t \in ]t_{n-1}, t_n] \\ 0 & \text{if } t \in ]t_n, 1] \end{cases}$$

from where the stated value of $[A]_i$, for every $i \geqslant 0$, is easily deduced. □

The next two corollaries are direct consequences of the description of the bracket cardinality provided by the last lemma. We leave their proofs to the reader.

**Corollary 3.** *For every $A \in FMS(]0,1])$ and for every $a \in ]0,1[$,*

*(i)* $[A]_i \leqslant a$ *if and only if* $i > \sum_{t>a} A(t)$.

*(ii)* $[A]_i < a$ *if and only if* $i > \sum_{t\geqslant a} A(t)$.

**Corollary 4.** *For every $A, B \in FMS(]0,1])$, if $[A] = [B]$, then $A = B$.*

The following technical lemma will be used henceforth several times.

**Lemma 5.** *Let $\mathcal{C} : FMS(]0,1]) \to \overline{\mathbb{N}}$ be a fuzzy cardinality. Let $A$ be a non-null finite multiset over $]0,1]$, say with $Supp(A) = \{t_1, \ldots, t_n\}$. Then, for every $k \in \mathbb{N}$,*

$$\mathcal{C}(A)(k) = \bigvee \Big\{ \bigwedge_{j=1}^{n} \mathcal{C}(1/t_j)(i_{j,1}) \wedge \cdots \wedge \mathcal{C}(1/t_j)(i_{j,A(t_j)}) \mid \sum_{j=1}^{n} \sum_{l=1}^{A(t_j)} i_{j,l} = k \Big\}.$$

*Proof.* Since $A$ decomposes into

$$A = \sum_{j=1}^{n} A(t_j)/t_j = \sum_{j=1}^{n} \overbrace{1/t_j + \cdots + 1/t_j}^{A(t_j)},$$

the additivity of $\mathcal{C}$ implies that

$$\mathcal{C}(A) = \bigoplus_{j=1}^{n} \overbrace{\mathcal{C}(1/t_j) \oplus \cdots \oplus \mathcal{C}(1/t_j)}^{A(t_j)}.$$

The expression in the statement is a direct consequence of this decomposition and equation 1 at the end of Section 2. □

As a first consequence of this lemma, we know the form of a general fuzzy cardinal of a singleton $n/1$.

**Corollary 6.** *For every $n, k \in \mathbb{N}$,*

$$\mathcal{C}(n/1)(k) = \begin{cases} \mathcal{C}(1/1)(0) & \text{if } k < n \\ 1 & \text{if } k = n \\ \mathcal{C}(1/1)(2) & \text{if } k > n \end{cases}$$

*Proof.* The case $n = 1$ is given by the variability and consistency properties. The case $n = 0$ and $k \geqslant 1$ is also given by the variability property. Now, to prove that $\mathcal{C}(0/1)(0) = 1$, notice that, from the additivity property and the fact that $1/1 + 0/1 = 1/1$, we deduce that

$$\begin{aligned} \mathcal{C}(1/1)(0) &= (\mathcal{C}(1/1) \oplus \mathcal{C}(0/1))\,(0) = \mathcal{C}(1/1)(0) \wedge \mathcal{C}(0/1)(0) \\ 1 = \mathcal{C}(1/1)(1) &= (\mathcal{C}(1/1) \oplus \mathcal{C}(0/1))\,(1) \\ &= (\mathcal{C}(1/1)(1) \wedge \mathcal{C}(0/1)(0)) \vee (\mathcal{C}(1/1)(0) \wedge \mathcal{C}(0/1)(1)) \\ &= \mathcal{C}(0/1)(0) \vee (\mathcal{C}(1/1)(0) \wedge \mathcal{C}(1/1)(2)) \end{aligned}$$

(in the last equality we have used that $\mathcal{C}(1/1) = 1$, by the consistency property, and that $\mathcal{C}(0/1)(1) = \mathcal{C}(1/1)(2)$, by the variability property). Now, still by the consistency property, $\mathcal{C}(1/1)(0)$ is 1 or 0. In the first case, the first equality becomes $1 = 1 \wedge \mathcal{C}(0/1)(0)$, which implies that $\mathcal{C}(0/1)(0) = 1$. In the second case, the second equality becomes $1 = \mathcal{C}(0/1)(0) \vee (0 \wedge \mathcal{C}(1/1)(2)) = \mathcal{C}(0/1)(0) \vee 0$, from where we deduce again that $\mathcal{C}(0/1)(0) = 1$.

Now assume that $n \geqslant 2$. By the previous lemma, we have that

$$\mathcal{C}(n/1)(k) = \bigvee \{\mathcal{C}(1/1)(i_1) \wedge \cdots \wedge \mathcal{C}(1/1)(i_n) \mid i_1 + \cdots + i_n = k\Big\}. \tag{2}$$

Since the only decomposition of 0 as a sum of natural numbers is as a sum of 0's, this equality implies that

$$\mathcal{C}(n/1)(0) = \overbrace{\mathcal{C}(1/1)(0) \wedge \cdots \wedge \mathcal{C}(1/1)(0)}^{n} = \mathcal{C}(1/1)(0).$$

Now, we shall distinguish between $\mathcal{C}(1/1)(i) = 0$ for every $i \geqslant 2$ or $\mathcal{C}(1/1)(i) = 1$ for every $i \geqslant 2$.

In the first case, it is clear that, for every $k = 1, \ldots, n-1$, all terms of the form

$$\mathcal{C}(1/1)(i_1) \wedge \cdots \wedge \mathcal{C}(1/1)(i_n)$$

with $i_1 + \cdots + i_n = k$ are 0 except

$$\overbrace{\mathcal{C}(1/1)(1) \wedge \cdots \wedge \mathcal{C}(1/1)(1)}^{k} \wedge \overbrace{\mathcal{C}(1/1)(0) \wedge \cdots \wedge \mathcal{C}(1/1)(0)}^{n-k} = 1 \wedge \mathcal{C}(1/1)(0) = \mathcal{C}(1/1)(0),$$

and hence, by (2), $\mathcal{C}(n/1)(k) = \mathcal{C}(1/1)(0)$.

As far as $\mathcal{C}(n/1)(n)$ goes, the decomposition $n = \overbrace{1 + \cdots + 1}^{n}$ yields

$$\overbrace{\mathcal{C}(1/1)(1) \wedge \cdots \wedge \mathcal{C}(1/1)(1)}^{n} = 1$$

and therefore, by (2), $\mathcal{C}(n/1)(n) = 1$.

Finally, every decomposition of any $k > n$ as $k = i_1 + \cdots + i_n$, with $i_1, \ldots, i_n \geqslant 0$, involves some summand $i_j \geqslant 2$, and then, being $\mathcal{C}(1/1)(i_j) = 0$, the corresponding

$$\mathcal{C}(1/1)(i_1) \wedge \cdots \wedge \mathcal{C}(1/1)(i_j) \wedge \cdots \wedge \mathcal{C}(1/1)(i_n)$$

is 0. This implies, still by (2), that $\mathcal{C}(n/1)(k) = 0 = \mathcal{C}(1/1)(2)$ for every $k > n$.

Consider now the second case, when $\mathcal{C}(1/1)(i) = 1$ for every $i \geqslant 2$. If $k < n$, every term of the form $\mathcal{C}(1/1)(i_1) \wedge \cdots \wedge \mathcal{C}(1/1)(i_n)$ with $i_1 + \cdots + i_n = k$ is the meet of some 1's and at least one $\mathcal{C}(1/1)(0)$ and hence it is equal to $\mathcal{C}(1/1)(0)$. This, again by (2), implies that $\mathcal{C}(n/1)(k) = \mathcal{C}(1/1)(0)$.

As in the previous case, $\overbrace{\mathcal{C}(1/1)(1) \wedge \cdots \wedge \mathcal{C}(1/1)(1)}^{n} = 1$ implies, still by (2), that $\mathcal{C}(n/1)(n) = 1$.

And finally, if $k > n$, then the decomposition $k = (k - n + 1) + \overbrace{1 + \cdots + 1}^{n-1}$ yields

$$\mathcal{C}(1/1)(k - n + 1) \wedge \overbrace{\mathcal{C}(1/1)(1) \wedge \cdots \wedge \mathcal{C}(1/1)(1)}^{n-1} = 1$$

and thus, by (2), $\mathcal{C}(n/1)(k) = 1 = \mathcal{C}(1/1)(2)$ for every $k > n$. □

**Remark 3.** Notice that, depending on the values of $\mathcal{C}(1/1)(0), \mathcal{C}(1/1)(2) \in \{0, 1\}$, there are four possibilities for the value of $\mathcal{C}(n/1)$:

- If $\mathcal{C}(1/1)(0) = \mathcal{C}(1/1)(2) = 0$, then

$$\mathcal{C}(n/1)(k) = \begin{cases} 1 & \text{if } k = n \\ 0 & \text{otherwise} \end{cases}$$

- If $\mathcal{C}(1/1)(0) = 1$ and $\mathcal{C}(1/1)(2) = 0$, then

$$\mathcal{C}(n/1)(k) = \begin{cases} 1 & \text{if } k \leqslant n \\ 0 & \text{otherwise} \end{cases}$$

- If $\mathcal{C}(1/1)(0) = 0$ and $\mathcal{C}(1/1)(2) = 1$, then

$$\mathcal{C}(n/1)(k) = \begin{cases} 1 & \text{if } k \geqslant n \\ 0 & \text{otherwise} \end{cases}$$

- If $\mathcal{C}(1/1)(0) = \mathcal{C}(1/1)(2) = 1$, then $\mathcal{C}(n/1)(k) = 1$ for every $k \in \mathbb{N}$.

The first three cases correspond to three natural ways of considering the natural number $n$ as a generalized natural number; and as we shall see below, in the fourth case $\mathcal{C}(A)$ will be the constant 1 mapping for every $A \in FMS(]0,1])$.

**Remark 4.** Arguing as in Remark 2, we obtain that if $\mathcal{C} : FMS(]0,1]) \to \overline{\mathbb{N}}$ is a fuzzy cardinality, then

$$\mathcal{C}(A \vee B) \oplus \mathcal{C}(A \wedge B) = \mathcal{C}(A) \oplus \mathcal{C}(B) \text{ for every } A, B \in FMS(]0,1]).$$

In particular, if $A \wedge B = \perp$, then $A + B = A \vee B$ and the additivity of fuzzy cardinalities implies that $\mathcal{C}(A \vee B) = \mathcal{C}(A) \oplus \mathcal{C}(B)$.

We also have the following result.

**Proposition 7.** *Let $\mathcal{C}$ be a fuzzy cardinality on $FMS(]0,1])$. If $A, B \in MS(]0,1])$ are such that $A \leqslant B$, then the equation*

$$\mathcal{C}(A) \oplus \alpha = \mathcal{C}(B)$$

*has a solution in $\overline{\mathbb{N}}$, and one such solution is $\mathcal{C}(B - A)$.*

*Proof.* Since $A + (B - A) = B$, the additivity of fuzzy cardinalities entails that $\mathcal{C}(A) \oplus \mathcal{C}(B - A) = \mathcal{C}(B)$. □

Our main result will establish that all fuzzy cardinalities on $FMS(]0,1])$ can be obtained in terms of the bracket cardinality in the way described by the following definition.

**Definition 3.** *Let $f : [0,1] \to [0,1]$ be an increasing mapping such that $f(0) \in \{0,1\}$ and $f(1) = 1$ and let $g : [0,1] \to [0,1]$ be a decreasing mapping such that $g(0) = 1$ and $g(1) \in \{0,1\}$.*

*Let $\mathcal{C}_{f,g} : FMS(]0,1]) \to \overline{\mathbb{N}}$ be the mapping defined as follows: for every $A \in FMS(]0,1])$ and $i \in \mathbb{N}$,*

$$\mathcal{C}_{f,g}(A)(i) = f([A]_i) \wedge g([A]_{i+1}).$$

This definition is correct because of the following lemma.

**Lemma 8.** *Let $f, g : [0,1] \to [0,1]$ be as in the last definition. Then, for every $A \in FMS(]0,1])$, the mapping $\mathcal{C}_{f,g}(A) : \mathbb{N} \to [0,1]$ is convex.*

*Proof.* Let $i \leqslant j \leqslant k$. Then, since $[A]$ is decreasing, $[A]_i \geqslant [A]_j \geqslant [A]_k$ and $[A]_{i+1} \geqslant [A]_{j+1} \geqslant [A]_{k+1}$, and therefore, since $f$ is increasing and $g$ is decreasing, $f([A]_i) \geqslant f([A]_j) \geqslant f([A]_k)$ and $g([A]_{i+1}) \leqslant g([A]_{j+1}) \leqslant g([A]_{k+1})$. This implies that

$$\begin{aligned}\mathcal{C}_{f,g}(A)(i) \wedge \mathcal{C}_{f,g}(A)(k) &= (f([A]_i) \wedge g([A]_{i+1})) \wedge (f([A]_k) \wedge g([A]_{k+1}))\\ &= (f([A]_i) \wedge f([A]_k)) \wedge (g([A]_{i+1}) \wedge g([A]_{k+1}))\\ &= f([A]_k) \wedge g([A]_{i+1}) \leqslant f([A]_j) \wedge g([A]_{j+1}) = \mathcal{C}_{f,g}(A)(j).\end{aligned}$$

Being $i, j, k \in \mathbb{N}$ arbitrary, this entails that $\mathcal{C}_{f,g}(A)$ is convex. □

**Remark 5.** If $f$ is the constant mapping 1, then, as we saw in the proof of the previous lemma,

$$\begin{array}{rccl}\mathcal{C}_{1,g}(A): & \mathbb{N} & \rightarrow & [0,1]\\ & i & \mapsto & 1 \wedge g([A]_{i+1}) = g([A]_{i+1})\end{array}$$

is an increasing mapping. In a similar way, if $g$ is the constant mapping 1, then

$$\begin{array}{rccl}\mathcal{C}_{f,1}(A): & \mathbb{N} & \rightarrow & [0,1]\\ & i & \mapsto & f([A]_i) \wedge 1 = f([A]_i)\end{array}$$

is a decreasing mapping. Finally, if $f$ and $g$ are both non-constant, then, for every $k \in \mathbb{N}$,

$$\begin{aligned}\mathcal{C}_{f,g}(A)(k) &= \mathcal{C}_{f,1}(A)(k) \wedge \mathcal{C}_{1,g}(A)(k+1)\\ &= \begin{cases}\mathcal{C}_{f,1}(A)(k) & \text{if } \mathcal{C}_{f,1}(A)(k) \leqslant \mathcal{C}_{1,g}(A)(k+1)\\ \mathcal{C}_{1,g}(A)(k+1) & \text{if } \mathcal{C}_{1,g}(A)(k+1) \leqslant \mathcal{C}_{f,1}(A)(k)\end{cases}\end{aligned}$$

Since $\mathcal{C}_{f,1}(A)$ is decreasing and $\mathcal{C}_{1,g}(A)$ is increasing, we have that if $\mathcal{C}_{1,g}(A)(k+1) \leqslant \mathcal{C}_{f,1}(A)(k)$ for some $k$, then $\mathcal{C}_{1,g}(A)(i+1) \leqslant \mathcal{C}_{f,1}(A)(i)$ for every $i \leqslant k$, and that if $\mathcal{C}_{f,1}(A)(k) \leqslant \mathcal{C}_{1,g}(A)(k+1)$ for some $k$, then $\mathcal{C}_{f,1}(A)(i) \leqslant \mathcal{C}_{1,g}(A)(i+1)$ for every $i \geqslant k$.

This implies that there exists an $n_0 \in \mathbb{N}$ such that $\mathcal{C}_{f,g}(A)$ is given by (the increasing mapping) $\mathcal{C}_{1,g}(A)$ on $\{i \in \mathbb{N} \mid i < n_0\}$ and by (the decreasing mapping) $\mathcal{C}_{f,1}(A)$ on $\{i \in \mathbb{N} \mid i \geqslant n_0\}$.

Now we have the following result.

**Theorem 9.** *A mapping $\mathcal{C} : FMS(]0,1]) \rightarrow \overline{\mathbb{N}}$ is a fuzzy cardinality if and only if $\mathcal{C} = \mathcal{C}_{f,g}$ for some increasing mapping $f : [0,1] \rightarrow [0,1]$ such that $f(0) \in \{0,1\}$ and $f(1) = 1$ and some decreasing mapping $g : [0,1] \rightarrow [0,1]$ such that $g(0) = 1$ and $g(1) \in \{0,1\}$. Moreover, if $\mathcal{C}_{f,g} = \mathcal{C}_{f',g'}$, then $f = f'$ and $g = g'$.*

In order no to loose the thread of the paper, we postpone the long proof of this theorem until an appendix at the end of the paper.

From now on, and to simplify the language, every time we speak about "the fuzzy cardinality $\mathcal{C}_{f,g}$," or an equivalent expression, we shall assume, usually without any further mention, that $f$ and $g$ are two mappings $[0,1] \to [0,1]$ satisfying the assumptions in Definition 3. We shall call this $\mathcal{C}_{f,g}$ the *fuzzy cardinality generated by $f$ and $g$.*

**Example 3.** Let $g$ be the constant mapping 1 and $f$ the identity Id on $[0,1]$. Then $\mathcal{C}_{\mathrm{Id},1}$ is the fuzzy cardinality defined by

$$\mathcal{C}_{\mathrm{Id},1}(A)(i) = [A]_i \text{ for every } A \in FMS(]0,1]) \text{ and } i \in \mathbb{N};$$

i.e., it is the bracket fuzzy cardinality [ ] defined in Example 2.

**Example 4.** Let $g$ be the constant mapping 1 and $f_a : [0,1] \to [0,1]$, with $a \in ]0,1[$, the mapping defined by $f_a(t) = 0$ for every $t < a$ and $f_a(t) = 1$ for every $t \geqslant a$. Then

$$\mathcal{C}_{f_a,1}(A)(i) = f_a([A]_i) = \begin{cases} 1 & \text{if } [A]_i \geqslant a \\ 0 & \text{if } [A]_i < a \end{cases}$$

Since, by Corollary 3, $[A]_i \geqslant a$ if and only if $i \leqslant \sum_{t \geqslant a} A(t)$, we have that

$$\mathcal{C}_{f_a,1}(A)(i) = \begin{cases} 1 & \text{if } i \leqslant \sum_{t \geqslant a} A(t) \\ 0 & \text{if } i > \sum_{t \geqslant a} A(t) \end{cases}$$

**Example 5.** Let $f$ be the constant mapping 1 and $g : [0,1] \to [0,1]$ the mapping $1 - \mathrm{Id}$, defined by $g(t) = 1 - t$ for every $t \in [0,1]$. Then

$$\mathcal{C}_{1,1-\mathrm{Id}}(A)(i) = 1 - [A]_{i+1} \quad \text{for every } A \in FMS(]0,1]) \text{ and every } i \in \mathbb{N}.$$

**Example 6.** Let $f$ to be the constant mapping 1 and $g_a : [0,1] \to [0,1]$, with $a \in ]0,1[$, the mapping defined by $g_a(t) = 1$ for every $t < a$ and $g_a(t) = 0$ for every $t \geqslant a$. Then

$$\mathcal{C}_{1,g_a}(A)(i) = g_a([A]_{i+1}) = \begin{cases} 0 & \text{if } [A]_{i+1} \geqslant a \\ 1 & \text{if } [A]_{i+1} < a \end{cases}$$

for every $A \in FMS(]0,1])$ and for every $i \in \mathbb{N}$. This cardinality is, roughly speaking, the increasing version of the fuzzy cardinality $\mathcal{C}_{f_a,1}$ in Example 4.

**Example 7.** Let $f$ be the identity Id on $[0,1]$ and $g = 1 - \mathrm{Id}$. Then

$$\mathcal{C}_{\mathrm{Id},1-\mathrm{Id}}(A)(i) = [A]_i \wedge (1 - [A]_{i+1}) \quad \text{for every } A \in FMS(]0,1]) \text{ and every } i \in \mathbb{N}.$$

To understand what this cardinality measures, let us first notice that $\mathcal{C}_{\mathrm{Id},1-\mathrm{Id}}(A)(i) = [A]_i$ if and only if $[A]_i \leqslant 1 - [A]_{i+1}$, i.e., if and only if $[A]_i + [A]_{i+1} \leqslant 1$.

So, let

$$n_0 = \min\{i \in \mathbb{N} \mid [A]_i \leqslant \frac{1}{2}\} = \sum_{t > \frac{1}{2}} A(t) + 1 \text{ (by Corollary 3).}$$

Then,

**If $\mathbf{i \leqslant n_0 - 2}$**, then $[A]_i > \frac{1}{2}$ and $[A]_{i+1} > \frac{1}{2}$, and hence $[A]_i + [A]_{i+1} > 1$. This implies that, in this case,

$$\mathcal{C}_{\mathrm{Id},1-\mathrm{Id}}(A)(i) = 1 - [A]_{i+1}.$$

**If $\mathbf{i = n_0 - 1}$**, then $[A]_{n_0-1} > \frac{1}{2}$ but $[A]_{n_0} \leqslant \frac{1}{2}$, and we don't know *a priori* whether $[A]_{n_0-1} + [A]_{n_0} \leqslant 1$ or not. Then, in this case we can only state that

$$\mathcal{C}_{\mathrm{Id},1-\mathrm{Id}}(A)(n_0 - 1) = [A]_{n_0-1} \wedge (1 - [A]_{n_0}).$$

**If $\mathbf{i \geqslant n_0}$**, then $[A]_i \leqslant \frac{1}{2}$ and, being $[A]$ decreasing, $[A]_{i+1} \leqslant \frac{1}{2}$, too. Therefore, $[A]_i + [A]_{i+1} \leqslant 1$, which implies that, in this case,

$$\mathcal{C}_{\mathrm{Id},1-\mathrm{Id}}(A)(i) = [A]_i.$$

Thus, the generalized natural number $\mathcal{C}_{\mathrm{Id},1-\mathrm{Id}}(A)$ is increasing on $\{0, \ldots, n_0 - 2\}$ and decreasing on $\{n_0, n_0 + 1, \ldots\}$, and it takes its greatest value at $n_0 - 2$ or at $n_0 - 1$, or in an interval containing one of these elements.

**Example 8.** Let $f : [0, 1] \to [0, 1]$ be the mapping defined by $f(t) = 0$ if $t \leqslant \frac{1}{4}$ and $f(t) = t$ if $t > \frac{1}{4}$, and let $g : [0, 1] \to [0, 1]$ be the mapping defined by $g(t) = 1 - 2t$ if $t \leqslant \frac{1}{2}$ and $g(t) = 0$ if $t \geqslant \frac{1}{2}$. To give a more explicit description of

$$\mathcal{C}_{f,g}(A) = f([A]_i) \wedge g([A]_{i+1}),$$

for a given $A \in FMS(]0, 1])$, let

$$\begin{aligned} n_A &= \min\{i \mid [A]_i < \tfrac{1}{2}\} = \textstyle\sum_{t \geqslant \frac{1}{2}} A(t) + 1 \\ i_A &= \min\{i \mid [A]_i < \tfrac{1}{4}\} = \textstyle\sum_{t \geqslant \frac{1}{4}} A(t) + 1; \end{aligned}$$

notice that $n_A \leqslant i_A$ and that

$$f([A]_i) = \begin{cases} [A]_i & \text{if } i < i_A \\ 0 & \text{if } i \geqslant i_A \end{cases} \qquad g([A]_{i+1}) = \begin{cases} 0 & \text{if } i < n_A - 1 \\ 1 - 2[A]_{i+1} & \text{if } i \geqslant n_A - 1 \end{cases}$$

Then,

$$\mathcal{C}_{f,g}(A)(i) = \begin{cases} 0 & \text{if } i < n_A - 1 \\ [A]_i \wedge (1 - 2[A]_{i+1}) & \text{if } n_A - 1 \leqslant i < i_A \\ 0 & \text{if } i \geqslant i_A \end{cases}$$

To analyze the behaviour of this mapping on the interval $n_A - 1 \leqslant i \leqslant i_A - 1$, notice that $\mathcal{C}_{f,g}(A)(i) = [A]_i$ if and only if $[A]_i \leqslant 1 - 2[A]_{i+1}$, i.e., if and only if $[A]_i + 2[A]_{i+1} \leqslant 1$. Then, if we let

$$m_A = \min\{i \in \mathbb{N} \mid [A]_i \leqslant \frac{1}{3}\} = \sum_{t > \frac{1}{3}} A(t) + 1$$

(and notice that $n_A \leqslant m_A \leqslant i_A$), and we argue as in the last example, we obtain finally that

$$\mathcal{C}_{f,g}(A) = \begin{cases} 0 & \text{if } i < n_A - 1 \\ 1 - 2[A]_{i+1} & \text{if } n_A - 1 \leqslant i < m_A - 1 \\ [A]_{m_A - 1} \wedge (1 - 2[A]_{m_A}) & \text{if } i = m_A - 1 \\ [A]_i & \text{if } m_A \leqslant i < i_A \\ 0 & \text{if } i \geqslant i_A \end{cases}$$

**Remark 6.** It is straightforward to prove from the explicit description of the bracket cardinal given in Lemma 2 that, for every increasing mapping $f : [0,1] \to [0,1]$ such that $f(0) \in \{0,1\}$ and $f(1) = 1$,

$$Sc_f(A) = \sum_{i=1}^{\infty} \mathcal{C}_{f,1}(A)(i), \quad \text{for every } A \in FMS(]0,1]).$$

We shall now prove that any fuzzy cardinality is the meet of an increasing and a decreasing fuzzy cardinalities.

**Definition 4.** *A fuzzy cardinality $\mathcal{C} : FMS(]0,1]) \to \overline{\mathbb{N}}$ is* increasing *(resp.,* decreasing*) if and only if $\mathcal{C}(A) \in \overline{\mathbb{N}}$ is an increasing (resp., decreasing) mapping, for every $A \in FMS(]0,1])$.*

**Proposition 10.** *For every fuzzy cardinality $\mathcal{C}_{f,g}$ on $FMS(]0,1])$, the following assertions are equivalent:*

*(i) $\mathcal{C}_{f,g}$ is increasing.*

*(ii) $f$ is the constant mapping $1$.*

*(iii) $\mathcal{C}_{f,g}(A)(k) = g([A]_{k+1})$ for every $A \in FMS(]0,1])$ and $k \in \mathbb{N}$.*

*Proof.* As far as the implication (i)$\Longrightarrow$(ii) goes, if $\mathcal{C}_{f,g}$ is an increasing fuzzy cardinality, then, in particular, $\mathcal{C}_{f,g}(1/1)$ is an increasing generalized natural number. Now, $\mathcal{C}_{f,g}(1/1)(1) = 1$ by the variability property, and hence $\mathcal{C}_{f,g}(1/1)(2) = 1$, too. Then,

$$1 = \mathcal{C}_{f,g}(1/1)(2) = f([1/1]_2) \wedge g([1/1]_3) = f(0) \wedge g(0) = f(0) \wedge 1 = f(0)$$

implies that $f(0) = 1$. Thus, since $f$ an increasing mapping, it must be the constant mapping 1.

As far as the implications (ii)$\Longrightarrow$(iii) and (iii)$\Longrightarrow$(i) go, see Remark 5. □

**Proposition 11.** *For every fuzzy cardinality $\mathcal{C}_{f,g}$ on $FMS(]0,1])$, the following assertions are equivalent:*

*i) $\mathcal{C}_{f,g}$ is a decreasing cardinality.*

*ii) g is the constant mapping 1.*

*(iii)* $\mathcal{C}_{f,g}(A)(k) = f([A]_k)$ *for every* $A \in FMS(]0,1])$ *and* $k \in \mathbb{N}$.

*Proof.* Assume that $\mathcal{C}_{f,g}$ is a decreasing fuzzy cardinality. Then, in particular, $\mathcal{C}_{f,g}(1/1)$ is a decreasing generalized natural number. Now, $\mathcal{C}_{f,g}(1/1)(1) = 1$ and

$$\mathcal{C}_{f,g}(1/1)(0) = f([1/1]_0) \wedge g([1/1]_1) = f(1) \wedge g(1) = 1 \wedge g(1) = g(1).$$

Therefore, $\mathcal{C}_{f,g}(1/1)(0) \geqslant \mathcal{C}_{f,g}(1/1)(1)$ implies $g(1) \geqslant 1$, i.e., $g(1) = 1$. And then, since $g$ is a decreasing mapping, it must be the constant mapping 1.

This proves the implication (i)$\Longrightarrow$(ii). As far as the implications (ii)$\Longrightarrow$(iii) and (iii)$\Longrightarrow$(i) go, see again Remark 5. □

**Remark 7.** Notice that the only fuzzy cardinality which is both decreasing and increasing is $\mathcal{C}_{1,1}$, which is constant: $\mathcal{C}_{1,1}(A)(k) = 1$ for every $A \in FMS(]0,1])$ and $k \in \mathbb{N}$.

**Corollary 12.** *Every fuzzy cardinality on* $FMS(]0,1])$ *is the meet of an increasing and a decreasing fuzzy cardinalities.*

*Proof.* As we saw in Remark 5, $\mathcal{C}_{f,g}(A) = \mathcal{C}_{f,1}(A) \wedge \mathcal{C}_{1,g}(A)$, for every $A \in FMS(]0,1])$, and $\mathcal{C}_{1,g}(A)$ is increasing and $\mathcal{C}_{f,1}(A)$ is decreasing. □

**Corollary 13.** *The meet of two fuzzy cardinalities on* $FMS(]0,1])$ *is again a fuzzy cardinality.*

*Proof.* Let $\mathcal{C}_{f,g}$ and $\mathcal{C}_{f',g'}$ be the fuzzy cardinalities associated to the mappings $f, g : [0,1] \to [0,1]$ and $f', g' : [0,1] \to [0,1]$, respectively. We have just proved that $\mathcal{C}_{f,g} = \mathcal{C}_{f,1} \wedge \mathcal{C}_{1,g}$ and $\mathcal{C}_{f',g'} = \mathcal{C}_{f',1} \wedge \mathcal{C}_{1,g'}$, and hence, by the associativity of the meet operation $\wedge$ in $\overline{\mathbb{N}}$, for every $A \in FMS(]0,1])$

$$\begin{aligned}(\mathcal{C}_{f,g} \wedge \mathcal{C}_{f',g'})(A) &= (\mathcal{C}_{f,1}(A) \wedge \mathcal{C}_{1,g}(A)) \wedge (\mathcal{C}_{f',1}(A) \wedge \mathcal{C}_{1,g'}(A)) \\ &= (\mathcal{C}_{f,1}(A) \wedge \mathcal{C}_{f',1}(A)) \wedge (\mathcal{C}_{1,g}(A) \wedge \mathcal{C}_{1,g'}(A)).\end{aligned} \tag{3}$$

Now, if $f, f' : [0,1] \to [0,1]$ are two increasing mappings such that $f(0), f'(0) \in \{0,1\}$ and $f(1) = f'(1) = 1$, then their meet

$$\begin{array}{rrcl} f \wedge f' : & [0,1] & \mapsto & [0,1] \\ & t & \mapsto & f(t) \wedge f'(t) \end{array}$$

is also an increasing mapping that sends 0 to either 0 or 1, and 1 to 1. And it is clear from the definition that $\mathcal{C}_{f,1}(A) \wedge \mathcal{C}_{f',1}(A) = \mathcal{C}_{f \wedge f',1}(A)$ for every $A \in FMS(]0,1])$.

In a similar way, if $g, g' : [0,1] \to [0,1]$ are two decreasing mappings such that $g(0) = g'(0) = 1$ and $g(1), g'(1) \in \{0,1\}$, then

$$\begin{array}{rrcl} g \wedge g' : & [0,1] & \mapsto & [0,1] \\ & t & \mapsto & g(t) \wedge g'(t) \end{array}$$

satisfies also these properties and $\mathcal{C}_{1,g} \wedge \mathcal{C}_{1,g'} = \mathcal{C}_{1,g\wedge g'}$.

Therefore, from (3) and these observations we deduce that

$$(\mathcal{C}_{f,g} \wedge \mathcal{C}_{f',g'})(A) = \mathcal{C}_{f\wedge f',1}(A) \wedge \mathcal{C}_{1,g\wedge g'}(A) = \mathcal{C}_{f\wedge f',g\wedge g'}(A) \text{ for every } A \in FMS(]0,1]),$$

and in particular that $\mathcal{C}_{f,g} \wedge \mathcal{C}_{f',g'}$ is the fuzzy cardinality generated by the mappings $f \wedge f'$ and $g \wedge g'$. □

**Remark 8.** It is interesting to point out that the join of two fuzzy cardinalities need not be a fuzzy cardinality; actually, it need not even take values in $\overline{\mathbb{N}}$. For instance, $\mathcal{C} = \mathcal{C}_{\mathrm{Id},1} \vee \mathcal{C}_{1,1-\mathrm{Id}}$ is defined by

$$\mathcal{C}(A)(i) = [A]_i \vee (1 - [A]_{i+1}) \quad \text{for every } A \in FMS(]0,1]) \text{ and } i \in \mathbb{N}.$$

Now, let $A$ be the finite multiset on $]0,1]$ with $Supp(A) = \{0.2, 0.5\}$ and defined on this support by $A(0.2) = A(0.5) = 1$. Then

$$[A]_0 = 1,\ [A]_1 = 0.5,\ [A]_2 = 0.2,\ [A]_i = 0 \text{ for every } i \geqslant 3$$

and hence

$$\begin{array}{ll} \mathcal{C}(A)(0) & = [A]_0 \vee (1 - [A]_1) = 1 \vee 0.5 = 1 \\ \mathcal{C}(A)(1) & = [A]_1 \vee (1 - [A]_2) = 0.5 \vee 0.8 = 0.8 \\ \mathcal{C}(A)(2) & = [A]_2 \vee (1 - [A]_3) = 0.2 \vee 1 = 1 \end{array}$$

Thus, $\mathcal{C}(A)$ is not convex.

To close this section, let us point out the following result.

**Proposition 14.** *Let $f : [0,1] \to [0,1]$ be an strictly increasing mapping such that $f(0) = 0$ and $f(1) = 1$, and let $g : [0,1] \to [0,1]$ be an strictly decreasing mapping such that $g(0) = 1$ and $g(1) = 0$. Then, for every $A, B \in FMS(]0,1])$, $\mathcal{C}_{f,g}(A) = \mathcal{C}_{f,g}(B)$ if and only if $A = B$.*

*Proof.* The "only if" implication is obvious. As far as the "if" implication goes, by Remark 5, for every $A \in FMS(]0,1])$ there exists some $n_A \in \mathbb{N}$ such that

$$\mathcal{C}_{f,g}(A)(i) = \begin{cases} g([A]_{i+1}) & \text{if } i < n_A \\ f([A]_i) & \text{if } i \geqslant n_A \end{cases}$$

If $\mathcal{C}_{f,g}(A) = \mathcal{C}_{f,g}(B)$, then we can take $n_A = n_B$ and then $g([A]_{i+1}) = g([B]_{i+1})$ for every $i < n_A$ and $f([A]_i) = f([B]_i)$ for every $i \geqslant n_A$. Since $f$ and $g$ are injective, this implies that $[A]_i = [B]_i$ for every $i \geqslant 1$. Since $[A]_0 = 1 = [B]_0$ by definition, the equality $[A]_i = [B]_i$ holds for every $i \in \mathbb{N}$. But then, by Lemma 4, this implies that $A = B$. □

Of course, from this proof we can also deduce that if $f$ is injective and $g$ is the constant mapping 1 or $g$ is injective and $f$ is the constant mapping 1, then it also happens that $\mathcal{C}_{f,g}(A) = \mathcal{C}_{f,g}(B)$ if and only if $A = B$. Therefore, it is not necessary the injectivity of both $f$ and $g$ for the thesis of the last proposition to hold.

## 5 Multisets defined by fuzzy sets

The goal of this section is to show that if we associate to a finite fuzzy set $F : X \to [0,1]$ the multiset $M_F : ]0,1] \to \mathbb{N}$ that counts, for every $t > 0$, the number of elements of $X$ where $F$ takes the value $t$, then the scalar cardinalities of $M_F$ defined in Section 3 generalize the scalar cardinalities of $F$ introduced axiomatically in [24], and the fuzzy cardinalities of $M_F$ defined in Section 4 are equivalent to the fuzzy cardinalities of $F$ introduced axiomatically in [10].

As we have mentioned, every fuzzy set $F : X \to [0,1]$ that is *finite*, in the sense that its *support*

$$Supp(F) = \{x \in X \mid F(x) \neq 0\}$$

is finite, defines in a natural way the finite multiset over $]0,1]$

$$\begin{array}{rccl} M_F : & ]0,1] & \to & \mathbb{N} \\ & t & \mapsto & |F^{-1}(t)| \end{array}$$

where $|\ |$ denotes the usual cardinality of a crisp set. Notice that if $F$ is finite and $X$ is infinite, then $|F^{-1}(0)|$ will be infinite, and hence $M_F$ cannot be defined in general on 0.

Let us recall now the scalar and fuzzy cardinalities of finite fuzzy sets.

- *Scalar cardinalities* [24]. Every increasing mapping $f : [0,1] \to [0,1]$ such that $f(0) = 0$ and $f(1) = 1$ *generates* the scalar cardinality $\widehat{Sc_f}$ defined as follows: for every finite fuzzy set $F$ on a set $X$,

$$\widehat{Sc_f}(F) = \sum_{x \in Supp(F)} f(F(x)).$$

  And all scalar cardinalities of finite fuzzy sets are obtained in this way.

- *Fuzzy cardinalities* [10]. Every pair of mappings $f, g : [0,1] \to [0,1]$ with $f$ increasing and such that $f(0) \in \{0,1\}$ and $f(1) = 1$, and $g$ decreasing and such that $g(0) = 1$ and $g(1) \in \{0,1\}$, *generate* the fuzzy cardinality $\widehat{\mathcal{C}_{f,g}}$ defined as follows: for every finite fuzzy set $F$ on a set $X$,

  $$\widehat{\mathcal{C}_{f,g}}(F)(i) = f([F]_i) \wedge g([F]_{i+1}) \text{ for every } i \in \mathbb{N},$$

  where now $[F]$ stands for the fuzzy cardinality of fuzzy sets proposed by Zadeh in [26]:

  $$[F]_i = \bigvee\{t \in [0,1] \mid |\{x \in X \mid F(x) \geqslant t\}| \geqslant i\} \quad \text{for every } i \in \mathbb{N}.$$

  And all scalar cardinalities of finite fuzzy sets are obtained in this way.

One immediately notes that for every $f : [0,1] \to [0,1]$ for which we define a scalar cardinality $\widehat{Sc_f}$ on fuzzy sets on $X$, we have defined a scalar cardinality $Sc_f$ on multisets over $]0,1]$, and that for every $f, g : [0,1] \to [0,1]$ for which we define a fuzzy cardinality $\widehat{\mathcal{C}_{f,g}}$ on fuzzy sets on $X$, we have also defined a fuzzy cardinality $\mathcal{C}_{f,g}$ on multisets over $]0,1]$. Next two results show the relations that there exist between each $\widehat{Sc_f}$ and the corresponding $Sc_f$, on the one hand, and between $\widehat{\mathcal{C}_{f,g}}$ and the corresponding $\mathcal{C}_{f,g}$, on the other hand.

**Proposition 15.** *Let $f : [0,1] \to [0,1]$ be an increasing mapping such that $f(0) = 0$ and $f(1) = 1$. Let $\widehat{Sc_f}$ be the scalar cardinality of finite fuzzy sets of $X$ generated by $f$ and $Sc_f$ the scalar cardinality of finite multisets over $]0,1]$ generated by $f$. Then, for every fuzzy set $F$ on $X$,*

$$\widehat{Sc_f}(F) = Sc_f(M_F).$$

*Proof.* A simple computation shows that

$$\begin{aligned} Sc_f(M_F) &= \textstyle\sum_{t \in Supp(M_F)} f(t) M_F(t) = \sum_{t \in Supp(M_F)} f(t)|F^{-1}(t)| \\ &= \textstyle\sum_{t \in F(X)-\{0\}} \overbrace{f(t) + \cdots + f(t)}^{|F^{-1}(t)|} = \sum_{x \in Supp(F)} f(F(x)) = \widehat{Sc_f}(F). \end{aligned}$$

□

**Proposition 16.** *Let $f : [0,1] \to [0,1]$ be an increasing mapping such that $f(0) \in \{0,1\}$ and $f(1) = 1$ and let $g : [0,1] \to [0,1]$ be a decreasing mapping such that $g(0) = 1$ and $g(1) \in \{0,1\}$. Let $\widehat{\mathcal{C}_{f,g}}$ be the fuzzy cardinality of finite fuzzy sets of $X$ generated by $f$ and $g$ and let $\mathcal{C}_{f,g}$ be the fuzzy cardinality of finite multisets over $]0,1]$ generated by $f$ and $g$. Then, for every fuzzy set $F$ on $X$,*

$$\widehat{\mathcal{C}_{f,g}}(F) = \mathcal{C}_{f,g}(M_F).$$

*Proof.* To begin with, notice that, for every $i \in \mathbb{N}$,

$$\begin{aligned}[M_F]_i &= \bigvee\{t \in [0,1] \mid \textstyle\sum_{t' \geqslant t} M_F(t') \geqslant i\} \\ &= \bigvee\{t \in [0,1] \mid \textstyle\sum_{t' \geqslant t} |F^{-1}(t')| \geqslant i\} \\ &= \bigvee\{t \in [0,1] \mid |\{x \in X \mid F(x) \geqslant t\}| \geqslant i\} = [F]_i\end{aligned}$$

Then, $\mathcal{C}_{f,g}(M_F)(i) = f([M_F]_i) \wedge g([M_F]_{i+1}) = f([F]_i) \wedge g([F]_{i+1}) = \widehat{\mathcal{C}_{f,g}}(F)(i)$. □

## 6 Scalar and fuzzy cardinalities of finite fuzzy multisets

Let us fix from now on a crisp set $X$. Let us recall that a finite fuzzy multiset over a set $X$ is a mapping $\overline{M} : X \to FMS(]0,1])$ such that

$$Supp(\overline{M}) = \{x \in X \mid \overline{M}(x) \neq \perp\}$$

is finite. We shall denote the set of all finite fuzzy multisets over $X$ by $\mathcal{FFMS}(X)$.

In this section we generalize the axiomatic notion of scalar and fuzzy cardinalities of crisp multisets to fuzzy multisets by imposing the additivity condition and to behave like a cardinality of crisp multisets on the fuzzy multisets whose support is a singleton. We shall then show that the additivity condition makes each (scalar or fuzzy) cardinality of finite fuzzy multisets $\overline{M} : X \to FMS(]0,1])$ to be the sum of (scalar or fuzzy) cardinalities applied to the finite crisp multisets $\overline{M}(x)$, $x \in Supp(\overline{M})$.

For every $x \in X$ and for every $M \in FMS(]0,1])$, we shall denote by $M/x$ the finite fuzzy multiset over $X$ defined by $\overline{M}(x) = M$ and $\overline{M}(y) = \perp$ for every $y \neq x$.

**Definition 5.** *A* scalar cardinality *on $\mathcal{FFMS}(X)$ is a mapping $\widetilde{Sc} : \mathcal{FFMS}(X) \to \mathbb{R}^+$ that satisfies the following conditions:*

*(i)* $\widetilde{Sc}(\overline{A} + \overline{B}) = \widetilde{Sc}(\overline{A}) + \widetilde{Sc}(\overline{B})$ *for every* $\overline{A}, \overline{B} \in \mathcal{FFMS}(X)$.

*(ii)* $\widetilde{Sc}((1/1)/x) = 1$ *for every* $x \in X$.

*A scalar cardinality $\widetilde{Sc}$ on $\mathcal{FFMS}(X)$ is* homogeneous *when it satisfies the following extra property:*

*(iii)* $\widetilde{Sc}(M/x) = \widetilde{Sc}(M/y)$ *for every* $x, y \in X$ *and* $M \in FMS(]0,1])$.

The thesis in Remarks 1 and 2 in Section 3 still hold for scalar cardinalities on $\mathcal{FFMS}(X)$, because they are direct consequences of the additivity property. In particular, $\widetilde{Sc}(\perp) = 0$ for every scalar cardinality $\widetilde{Sc}$ on $\mathcal{FFMS}(X)$.

Next proposition provides a description of all scalar cardinalities on $\mathcal{FFMS}(X)$.

**Proposition 17.** *A mapping* $\widetilde{Sc} : \mathcal{FFMS}(X) \to \mathbb{R}^+$ *is a scalar cardinality if and only if for every* $x \in X$ *there exists an scalar cardinality* $Sc_x$ *on* $FMS(]0,1])$ *such that*

$$\widetilde{Sc}(\overline{M}) = \sum_{x \in X} Sc_x(\overline{M}(x)).$$

*Moreover, the family* $(Sc_x)_{x \in X}$ *is uniquely determined by* $\widetilde{Sc}$*, and* $\widetilde{Sc}$ *is homogeneous if and only if* $Sc_x = Sc_y$ *for every* $x, y \in X$*.*

*Proof.* Let $\widetilde{Sc}$ be a scalar cardinality on $\mathcal{FFMS}(X)$, and consider, for every $x \in X$, the mapping

$$\begin{array}{rrcl} Sc_x : & FMS(]0,1]) & \to & \mathbb{R}^+ \\ & M & \mapsto & \widetilde{Sc}(M/x) \end{array}$$

Conditions (i) and (ii) in Definition 5 entail that each $Sc_x$ satisfy conditions (i) and (ii) in Definition 1:

$$\begin{aligned} Sc_x(M_1 + M_2) &= \widetilde{Sc}((M_1 + M_2)/x) = \widetilde{Sc}(M_1/x + M_2/x) \\ &= \widetilde{Sc}(M_1/x) + \widetilde{Sc}(M_2/x) = Sc_x(M_1) + Sc_x(M_2) \\ Sc_x(1/1) &= \widetilde{Sc}((1/1)/x) = 1 \end{aligned}$$

Therefore, each $Sc_x$ is a scalar cardinality on $FMS(]0,1])$. Now, it is straightforward to check that, for every $\overline{M} \in \mathcal{FFMS}(X)$,

$$\overline{M} = \sum_{x \in Supp(\overline{M})} \overline{M}(x)/x.$$

Thus, the additivity property of $\widetilde{Sc}$ implies that

$$\widetilde{Sc}(\overline{M}) = \sum_{x \in Supp(\overline{M})} \widetilde{Sc}(\overline{M}(x)/x) = \sum_{x \in Supp(\overline{M})} Sc_x(\overline{M}(x)) = \sum_{x \in X} Sc_x(\overline{M}(x)).$$

And notice that if $\widetilde{Sc}$ is homogeneous, then $Sc_x = Sc_y$ for every $x, y \in X$ by definition.

Conversely, for every $x \in X$ let $Sc_x : FMS(]0,1]) \to \mathbb{R}^+$ be a scalar cardinality, and let $\widetilde{Sc} : \mathcal{FFMS}(X) \to \mathbb{R}^+$ be the mapping defined by

$$\widetilde{Sc}(\overline{M}) = \sum_{x \in X} Sc_x(\overline{M}(x))$$

for every $\overline{M} \in \mathcal{FFMS}(X)$; since $\overline{M}(x) = 0$ fo all $x \in X$ except for a finite number of them, this sum is well-defined. This mapping satisfies the defining conditions of scalar cardinalities on $\mathcal{FFMS}(]0,1])$:

(i) For every $\overline{A}, \overline{B} \in \mathcal{FFMS}(X)$,

$$\begin{aligned}
\widetilde{Sc}(\overline{A}+\overline{B}) &= \textstyle\sum_{x\in X} Sc_x((\overline{A}+\overline{B})(x)) \\
&= \textstyle\sum_{x\in X} Sc_x(\overline{A}(x)+\overline{B}(x)) \\
&= \textstyle\sum_{x\in X} (Sc_x(\overline{A}(x)) + Sc_x(\overline{B}(x))) \quad \text{(by Definition 1.(i))} \\
&= \widetilde{Sc}(\overline{A}) + \widetilde{Sc}(\overline{B})
\end{aligned}$$

(ii) $\widetilde{Sc}((1/1)/x) = Sc_x(1/1) = 1$ by Definition 1.(ii).

Finally, notice that

$$\widetilde{Sc}(M/x) = \sum_{y\in X} Sc_y((M/x)(y)) = Sc_x(M) + \sum_{y\neq x} Sc_y(\perp) = Sc_x(M),$$

which, together with the "only if" implication proved above, implies that every $Sc_x$ is uniquely determined by $\widetilde{Sc}$. And in particular, if $Sc_x = Sc_y$ for every $x, y \in M$, then $\widetilde{Sc}$ is homogeneous. □

Proposition 1 provides the following characterization of scalar cardinalities of fuzzy multisets.

**Corollary 18.** *(a) A mapping $\widetilde{Sc} : \mathcal{FFMS}(X) \to \mathbb{R}^+$ is a scalar cardinality if and only if for every $x \in X$ there exists a mapping $f_x : ]0,1] \to \mathbb{R}^+$ with $f_x(1) = 1$ such that*

$$\widetilde{Sc}(\overline{M}) = \sum_{x\in X} \sum_{t\in Supp(\overline{M}(x))} f_x(t)\overline{M}(x)(t) \qquad \text{for every } \overline{M} \in \mathcal{FFMS}(X).$$

*(b) A mapping $\widetilde{Sc} : \mathcal{FFMS}(X) \to \mathbb{R}^+$ is a homogeneous scalar cardinality if and only there exists a mapping $f : ]0,1] \to \mathbb{R}^+$ with $f(1) = 1$, such that*

$$\widetilde{Sc}(\overline{M}) = Sc_f(\sum_{x\in X} \overline{M}(x)) = \sum_{x\in X} \sum_{t\in Supp(\overline{M}(x))} f(t)\overline{M}(x)(t) \quad \text{for every } \overline{M} \in \mathcal{FFMS}(X).$$

Let us consider now the fuzzy cardinalities.

**Definition 6.** *A* fuzzy cardinality *on $\mathcal{FFMS}(X)$ is a mapping $\widetilde{\mathcal{C}} : \mathcal{FFMS}(X) \to \overline{\mathbb{N}}$ that satisfies the following conditions:*

*(i) For every $\overline{A}, \overline{B} \in \mathcal{FFMS}(X)$, $\widetilde{\mathcal{C}}(\overline{A}+\overline{B}) = \widetilde{\mathcal{C}}(\overline{A}) \oplus \widetilde{\mathcal{C}}(\overline{B})$.*

*(ii) For every $x \in X$, the mapping*

$$\begin{array}{rrcl}
\widetilde{\mathcal{C}}(\ /x) : & FMS(]0,1]) & \to & \overline{\mathbb{N}} \\
& M & \mapsto & \widetilde{\mathcal{C}}(M/x)
\end{array}$$

*is a fuzzy cardinality on $FM(]0,1])$*

*A fuzzy cardinality* $\widetilde{\mathcal{C}}$ *is* homogeneous *when it satisfies the following further condition:*

*(iii) For every* $x, y \in X$, $\widetilde{\mathcal{C}}(\ /x) = \widetilde{\mathcal{C}}(\ /y)$.

A simple argument, similar to the proof of Proposition 17, and which we leave to the reader, proves the following result.

**Proposition 19.** *A mapping* $\widetilde{\mathcal{C}} : \mathcal{FFMS}(X) \to \overline{\mathbb{N}}$ *is a fuzzy cardinality if and only if for every* $x \in X$ *there exists an fuzzy cardinality* $\mathcal{C}_x$ *on* $FMS(]0,1])$ *such that*

$$\widetilde{\mathcal{C}}(\overline{M}) = \bigoplus_{x \in X} \mathcal{C}_x(\overline{M}(x)).$$

*Moreover, the family* $(\mathcal{C}_x)_{x\in X}$ *is uniquely determined by* $\widetilde{\mathcal{C}}$, *and* $\widetilde{\mathcal{C}}$ *is homogeneous if and only if* $\mathcal{C}_x = \mathcal{C}_y$ *for every* $x, y \in X$.

Using Theorem 9, this proposition can be rewritten in the following way.

**Corollary 20.** *(a) A mapping* $\widetilde{\mathcal{C}} : \mathcal{FFMS}(X) \to \overline{\mathbb{N}}$ *is a fuzzy cardinality if and only if for every* $x \in X$ *there exist mappings* $f_x, g_x : [0,1] \to [0,1]$ *satisfying the hypothesis of Definition 3 such that*

$$\widetilde{\mathcal{C}}(\overline{M}) = \bigoplus_{x \in X} \mathcal{C}_{f_x,g_x}(\overline{M}(x)).$$

*(b) A mapping* $\widetilde{\mathcal{C}} : \mathcal{FFMS}(X) \to \overline{\mathbb{N}}$ *is a homogeneous fuzzy cardinality if and only if there exist mappings* $f, g : [0,1] \to [0,1]$ *satisfying the hypothesis of Definition 3 such that*

$$\widetilde{\mathcal{C}}(\overline{M}) = \mathcal{C}_{f,g}(\sum_{x\in X} \overline{M}(x)) = \bigoplus_{x \in X} \mathcal{C}_{f,g}(\overline{M}(x)).$$

Thus, homogeneous scalar and fuzzy cardinalities understand fuzzy multisets as a sum of crisp multisets, one for every type $x \in X$, and "count" this sum. Arbitrary scalar and fuzzy cardinalities "count" each multiset on each $x \in X$, possibly using a different cardinality for every $x \in X$, and then add up these results.

**Example 9.** Let $X = \{x_1, \ldots, x_n\}$. Then, the mapping $\widetilde{\mathcal{C}} : \mathcal{FFMS}(X) \to \overline{\mathbb{N}}$ defined by

$$\widetilde{\mathcal{C}}(\overline{M}) = [\overline{M}(x_1) + \cdots + \overline{M}(x_n)] = [\overline{M}(x_1)] \oplus \cdots \oplus [\overline{M}(x_n)]$$

is a homogeneous fuzzy cardinality on $\mathcal{FFMS}(X)$. This is the fuzzy cardinality of fuzzy multisets used by D. Rocacher in [19, §4.1].

**Example 10.** Let $X = \{x_1, \ldots, x_n\}$. Then, the mapping $\widetilde{\mathcal{C}} : \mathcal{FFMS}(X) \to \overline{\mathbb{N}}$ defined by

$$\widetilde{\mathcal{C}}(\overline{M}) = \mathcal{C}_{\mathrm{Id},1-\mathrm{Id}}(M_{x_1} + \cdots + M_{x_n}) = \mathcal{C}_{\mathrm{Id},1-\mathrm{Id}}(M_{x_1}) \oplus \cdots \oplus \mathcal{C}_{\mathrm{Id},1-\mathrm{Id}}(M_{x_n})$$

is a homogeneous fuzzy cardinality on $\mathcal{FFMS}(X)$.

**Example 11.** Let $X = \{x_1, \ldots, x_n\}$. For every $i = 1, \ldots, n$, let $f_i : [0,1] \to [0,1]$ denote the mapping defined by

$$f_i(t) = 0 \text{ if } t \leqslant \frac{1}{i+3} \text{ and } f_i(t) = t \text{ if } t > \frac{1}{i+3},$$

let $g_i : [0,1] \to [0,1]$ denote the mapping defined by

$$g_i(t) = 1 - (i+1)t \text{ if } t \leqslant \frac{1}{i+1)} \text{ and } g_i(t) = 0 \text{ if } t \geqslant \frac{1}{i+1},$$

and let $\mathcal{C}_i$ denote the fuzzy cardinality $\mathcal{C}_{f_i, g_i}$ on $FMS(]0,1])$. These cardinalities are similar to the one studied in Example 8. Then, the mapping $\widetilde{\mathcal{C}} : \mathcal{FFMS}(X) \to \overline{\mathbb{N}}$ defined by

$$\widetilde{\mathcal{C}}(\overline{M}) = \bigoplus_{i=1}^{n} \mathcal{C}_i(\overline{M}(x_i))$$

is a fuzzy cardinality on $\mathcal{FFMS}(X)$ that is not homogeneous: the contribution of each type $x_i$ to the multiset is measured through a different cardinality.

# 7 Conclusion

In this paper we have proposed axiomatic definitions for scalar and fuzzy cardinalities of finite fuzzy multisets over a set $X$. We have also characterized the resulting mappings from the set $\mathcal{FFMS}(X)$ of all finite fuzzy multisets over $X$ to $\mathbb{R}^+$ and to the set $\overline{\mathbb{N}}$ of generalized natural numbers, respectively, by means of simple constructions. The axiomatic definitions and the resulting characterizations are similar in flavour to those already known for cardinalities of fuzzy sets: cf. [24] and [10], respectively. And the families of cardinalities obtained through these axiomatic definitions contain as particular cases cardinalities of fuzzy multisets that had been previously introduced in the literature, like the usual scalar cardinality of fuzzy bags, which corresponds to our scalar cardinality $Sc_1$, and Rocacher's decreasing fuzzy cardinality $|\ |$, which is equal to our basic bracket fuzzy cardinality.

We have used the additivity property as the basic axiom in our definitions. Other properties can be used to replace this one. For instance, one could impose on the cardinal of the ordinary join of two fuzzy multisets to be the extended join of their cardinals as an alternative axiom, which would lead to a different family of axioms.

**Acknowledgements.** We thank the referees for their comments and suggestions, which have led to a substantial improvement of this paper.

# References

[1] L.l. Baowen, Fuzzy bags and applications. *Fuzzy Sets and Systems* 34 (1990), 61–72.

[2] W. D. Blizard, Multiset Theory. *Notre Dame J. Formal Logic* 30, (1989), 36–66.

[3] R. Biswas, An application of Yager's Bag Theory. *Int. J. Int. Syst.* 14 (1999), 231–1238.

[4] P. Bosc, D. Rocacher, About difference operation on fuzzy bags. *Proceedings IPMU 2002*, 1541–1546.

[5] P. Bosc, D. Rocacher, About Zf, the Set of Fuzzy Relative Integers, and the Definition of Fuzzy Bags on Zf. *Proceedings IFSA2003*, 95–102.

[6] J. Casasnovas, A solution for the division of a generalized natural number. *Proceedings of IPMU 2000*, 1583–1560

[7] J. Casasnovas, Cardinalidades escalares para divisores de cardinalidades difusas. *Actas del X congreso español sobre tecnologías y lògica fuzzy* (2000), 139–144

[8] J. Casasnovas, Scalar equipotency and fuzzy bijections. *Proceedings of EUSFLAT2001* (2001).

[9] J. Casasnovas, J. Miró, J. Moyà, F. Rosselló, An approach to membrane computing under uncertainty. *Int. J. Found. Comp. Sc.* 15 (2004), 841–864.

[10] J. Casasnovas, J.Torrens, An axiomatic approach to the fuzzy cardinality of finite fuzzy sets. *Fuzzy Sets and Systems* 133 (2003), 193–209.

[11] J. Casasnovas, J.Torrens, Scalar cardinalities of finite fuzzy sets for t-norms and t-conorms. *International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems* 11 (2003), 599–615.

[12] F. Connan, *Interrogation flexible de bases de données multimédias.* PhD Thesis, Univ. de Rennes I (1999).

[13] M. Delgado, D. Sánchez, M. J. Martín-Bautista, M.A. Vila, A probabilistic definition of a nonconvex fuzzy cardinality. *Fuzzy Sets and Systems* 126 (2002) 177–190.

[14] M. Delgado, M. J. Martín-Bautista, D. Sánchez, M. A. Vila Miranda, On a Characterization of Fuzzy Bags. *Proceedings IFSA2003*, 119–126.

[15] D. Dubois, A new definition of the fuzzy cardinality of finite sets preserving the classical additivity property, *Bull. Stud. Ecxch. Fuzziness Appl.(BUSEFAL)* 5 (1981) 11–12.

[16] Li Hong-xing, Luo Cheng-zhong and Wang Pei-zhuang, The cardinality of fuzzy sets and the continuum hipotesis. *Fuzzy Sets and Systems* 55 (1993) 61-78.

[17] Gh. Păun, *Membrane Computing. An Introduction.* Springer-Verlag (2002).

[18] D. Ralescu, Cardinality, quantifiers, and the aggregation of fuzzy criteria. *Fuzzy Sets and Systems* 69 (1995) 355–365.

[19] D. Rocacher, On fuzzy bags and their application to flexible querying. *Fuzzy Sets and Systems* 140 (2003), 93–110.

[20] A. Tzouvaras, Worlds of homogeneous artifacts. *Notre Dame J. Formal Logic* 36 (1995), 454–474.

[21] M. Wygralak, Vagueness and cardinality: A unifying approach. In *Fuzzy Logic and Soft Computing*, World Scientific (1995), 210–219.

[22] M. Wygralak, *Vaguely defined objects, Representations, fuzzy sets and nonxlassical cardinality theory*. Kluwer Academic Press (1996).

[23] M. Wygralak, Questions of cardinality of finite fuzzy sets. *Fuzzy Sets and Systems* 102 (1999) 185–210.

[24] M. Wygralak, An axiomatic approach to scalar cardinalities of fuzzy sets, *Fuzzy Sets and Systems* 110 (2000) 175-179.

[25] R. R. Yager, On the theory of bags. *Int. J. of General Systems* 13 (1986), 23–37.

[26] L. A. Zadeh, A computational approach to fuzzy quantifiers in natural languages. *Comput. Math. Appl.* 9 (1983) 149–184.

# Proof of Theorem 9

Recall that, given an increasing mapping $f : [0,1] \to [0,1]$ such that $f(0) \in \{0,1\}$ and $f(1) = 1$ and a decreasing mapping $g : [0,1] \to [0,1]$ such that $g(0) = 1$ and $g(1) \in \{0,1\}$, the mapping $\mathcal{C}_{f,g} : FMS(]0,1]) \to \overline{\mathbb{N}}$ is defined as follows: for every $A \in FMS(]0,1])$ and $i \in \mathbb{N}$,

$$\mathcal{C}_{f,g}(A)(i) = f([A]_i) \wedge g([A]_{i+1}).$$

The goal of this appendix is to prove the following result.

**Theorem 9** *A mapping* $\mathcal{C} : FMS(]0,1]) \to \overline{\mathbb{N}}$ *is a fuzzy cardinality if and only if* $\mathcal{C} = \mathcal{C}_{f,g}$ *for some pair of mappings* $f, g : [0,1] \to [0,1]$ *as above. Moreover, if* $\mathcal{C}_{f,g} = \mathcal{C}_{f',g'}$*, then* $f = f'$ *and* $g = g'$*.*

To ease the task of the reader, we split the proof into several steps.

**(I) An alternative expresion for $\mathcal{C}_{f,g}$.** We prove in this point that the mapping $\mathcal{C}_{f,g}$ can also be described recursively by the following rules:

- $\mathcal{C}_{f,g}(\perp)(0) = 1$ and $\mathcal{C}_{f,g}(\perp)(i) = f(0)$ for every $i \geqslant 1$;
- for every $t \in ]0, 1]$,

$$\mathcal{C}_{f,g}(1/t)(0) = g(t),\ \mathcal{C}_{f,g}(1/t)(1) = f(t),\ \mathcal{C}_{f,g}(1/t)(i) = f(0) \text{ for every } i \geqslant 2;$$

- for every $A \in FMS(]0, 1])$, $A \neq \perp$,

$$\mathcal{C}_{f,g}(A) = \bigoplus_{t \in Supp(A)} \overbrace{\mathcal{C}_{f,g}(1/t) \oplus \cdots \oplus \mathcal{C}_{f,g}(1/t)}^{A(t)}.$$

To simplify the notations, throughout this point, and once fixed the mappings $f$ and $g$, we shall denote $\mathcal{C}_{f,g}$ by $\mathcal{C}$.

For $\perp$ we have that

$$\begin{array}{l} \mathcal{C}(\perp)(0) = f([\perp]_0) \wedge g([\perp]_1) = f(1) \wedge g(0) = 1 \\ \mathcal{C}(\perp)(i) = f([\perp]_i) \wedge g([\perp]_{i+1}) = f(0) \wedge g(0) = f(0) \text{ for every } i \geqslant 1. \end{array}$$

For singletons $1/t$, we have that

$$\begin{array}{l} \mathcal{C}(1/t)(0) = f([1/t]_0) \wedge g([1/t]_1) = f(1) \wedge g(t) = g(t) \\ \mathcal{C}(1/t)(1) = f([1/t]_1) \wedge g([1/t]_2) = f(t) \wedge g(0) = f(t) \\ \mathcal{C}(1/t)(i) = f([1/t]_i) \wedge g([1/t]_{i+1}) = f(0) \wedge g(0) = f(0) \text{ for every } i \geqslant 2 \end{array}$$

Let finally $A$ be an arbitrary non-null multiset, say with support $Supp(A) = \{t_1, \ldots, t_n\} \neq \emptyset$, $t_1 < \cdots < t_n$. From the explicit description of the bracket cardinal given Lemma 2, we have that

$$f([A]_i) \wedge g([A]_{i+1}) = \begin{cases} g(t_n) & \text{if } i = 0 \\ f(t_n) \wedge g(t_n) & \text{if } 0 < i < A(t_n) \\ f(t_n) \wedge g(t_{n-1}) & \text{if } i = A(t_n) \\ f(t_{n-1}) \wedge g(t_{n-1}) & \text{if } A(t_n) < i < A(t_{n-1}) + A(t_n) \\ \vdots & \\ f(t_{s+1}) \wedge g(t_s) & \text{if } i = \sum_{j=s+1}^n A(t_j) \\ f(t_s) \wedge g(t_s) & \text{if } \sum_{j=s+1}^n A(t_j) < i < \sum_{j=s}^n A(t_j) \\ f(t_s) \wedge g(t_s) & \text{if } i = \sum_{j=s}^n A(t_j) \\ \vdots & \\ f(t_1) \wedge g(t_1) & \text{if } \sum_{j=2}^n A(t_j) < i < \sum_{j=1}^n A(t_j) \\ f(t_1) & \text{if } i = \sum_{j=1}^n A(t_j) \\ 0 & \text{if } \sum_{j=1}^n A(t_j) < i \end{cases}$$

Now, set

$$S(A) = \bigoplus_{j=1}^{n} \overbrace{\mathcal{C}_{f,g}(1/t_j) \oplus \cdots \oplus \mathcal{C}_{f,g}(1/t_j)}^{A(t_j)}.$$

We want to prove that $S(A) = \mathcal{C}(A)$ for every such multiset $A$. Recall that, by Lemma 5, for every $i \in \mathbb{N}$,

$$S(A)(i) = \bigvee \Big\{ \bigwedge_{j=1}^{n} \mathcal{C}(1/t_j)(i_{j,1}) \wedge \cdots \wedge \mathcal{C}(1/t_j)(i_{j,A(t_j)}) \mid \sum_{j=1}^{n} \sum_{l=1}^{A(t_j)} i_{j,l} = i \Big\};$$

in other words, to obtain $S(A)(i)$ one must compute the value of the meet

$$\bigwedge_{j=1}^{n} \mathcal{C}(1/t_j)(i_{j,1}) \wedge \cdots \wedge \mathcal{C}(1/t_j)(i_{j,A(t_j)}) \tag{4}$$

for every decomposition of $i$ as the sum of $Sc_1(A)$ natural numbers

$$i = i_{1,1} + \cdots + i_{1,A(t_1)} + i_{2,1} + \cdots + i_{j-1,A(t_{j-1})} + i_{j,1} + \cdots + i_{j,A(t_j)}, \tag{5}$$

and then find the greatest such value.

We shall distinguish two cases, depending on whether $f(0) = 0$ or $f(0) = 1$.

**(I.a) $\mathbf{f(0) = 0}$.** From the description of $\mathcal{C}(1/t_j)$ that we have just given, if $f(0) = 0$, then expression (4) is equal to 0 whenever some $i_{j,l}$ is greater or equal than 2. Therefore, to compute $S(A)(i)$ it is enough to consider decompositions (5) of $i$ with every $i_{j,l} \leqslant 1$. Now, for every such decomposition of $i$, expression (4) will be equal to

$$f(t_{j_1}) \wedge g(t_{j_2}),$$

where $j_1$ is the lowest index $j$ such that some $i_{j,l}$ is 1, and $j_2$ is the highest index $j$ such that some $i_{j,l}$ is 0; if every $i_{j,l}$ is 1, then (4) will be equal to $f(t_1)$, and if every $i_{j,l}$ is 0, then (4) will be equal to $g(t_n)$.

Let us check now that $S(A)(i) = f([A]_i) \wedge g([A]_{i+1})$ for every $i \in \mathbb{N}$ by dividing $\mathbb{N}$ into the same intervals as in the explicit description of the values $f([A]_i) \wedge g([A]_{i+1})$ given above.

- If $\sum_{j=1}^{n} A(t_j) < i$, then every decomposition (5) of $i$ involves some $i_{j,l} \geqslant 2$. As we have just pointed out, this implies that $S(A)(i) = 0$.
- If $i = \sum_{j=1}^{n} A(t_j)$, then the only decomposition (5) of $i$ that does not involve any $i_{j,l} \geqslant 2$ is the one with all summands 1. For this decomposition, as we have just mentioned, the expression (4) will be equal to $f(t_1)$. This implies that $S(A)(i) = f(t_1)$.

- If $\sum_{j=2}^{n} A(t_j) < i < \sum_{j=1}^{n} A(t_j)$, then there exists a decomposition (5) of $i$ such that $i_{j,l} = 1$, for every $j \geqslant 2$ and for every $l$, and there are $l_1, l_2$ such that $i_{1,l_1} = 1$ and $i_{1,l_2} = 0$. For this decomposition, the expression (4) is equal to $f(t_1) \wedge g(t_1)$.

  Besides, any decomposition of $i$ into 0s and 1s that does not have this form will have some $i_{j,l} = 0$ with $j \geqslant 2$, and for such a decomposition, the expressions (4) will be equal to $f(t_1) \wedge g(t_j)$. Since $g$ is decreasing, $f(t_1) \wedge g(t_1)$ is greater than all these other outcomes, and hence the value of $S(A)(i)$.

- If $i = \sum_{j=2}^{n} A(t_j)$, then we can decompose $i$ as in (5) with $i_{j,l} = 1$ for every $j \geqslant 2$ and every $l$, and $i_{1,l} = 0$ for every $l$. For this decomposition, the expression (4) is equal to $f(t_2) \wedge g(t_1)$.

  Besides, any other decomposition of $i$ into 0s and 1s will have some $i_{j,l} = 0$ with $j \geqslant 2$ and some $i_{1,l'} = 1$, and for such a decomposition the expression (4) will be equal to $f(t_1) \wedge g(t_j)$ with $j \geqslant 2$. Since $f$ is increasing and $g$ is decreasing, $f(t_2) \wedge g(t_1)$ is greater than all these other outcomes, and hence the value of $S(A)(i)$.

- In general, if $\sum_{j=s+1}^{n} A(t_j) < i < \sum_{j=s}^{n} A(t_j)$ for some $s = 1, \ldots, n$, then there exists a decomposition (5) of $i$ such that $i_{j,l} = 1$ for every $j > s$, $i_{j,l} = 0$ for every $j < s$, and there are $l_1, l_2$ such that $i_{s,l_1} = 1$ and $i_{s,l_2} = 0$. For this decomposition, the expression (4) is equal to $f(t_s) \wedge g(t_s)$.

  Besides, any decomposition of $i$ into 0s and 1s that does not have this form will use either some $i_{j,l} = 0$ with $j > s$ or some $i_{k,l} = 1$ with $k < s$, and for such a decomposition the meet (4) will be equal to $f(t_k) \wedge g(t_j)$, either with $k < s$ and $j \geqslant s$ or with $k \leqslant s$ and $j > s$. Since $f$ is increasing and $g$ is decreasing, $f(t_s) \wedge g(t_s)$ is greater than all these other outcomes, and hence the value of $S(A)(i)$.

- In general, if $i = \sum_{j=s}^{n} A(t_j)$ for some $s = 1, \ldots, n$, then we can decompose $i$ as in (5) with $i_{j,l} = 1$ for every $j \geqslant s$ and $i_{j,l} = 0$ for every $j < s$. For this decomposition, the expression (4) is equal to $f(t_s) \wedge g(t_{s-1})$.

  Besides, any other decomposition of $i$ into 0s and 1s will have some $i_{j,l} = 0$ with $j \geqslant s$ and some $i_{k,l} = 1$ with $k < s$, and for such a decomposition the expression (4) will be equal to $f(t_k) \wedge g(t_j)$ with $k < s$ and $j \geqslant s$. Since $f$ is increasing and $g$ is decreasing, $f(t_s) \wedge g(t_{s-1})$ is greater than all these other outcomes, and hence the value of $S(A)(i)$.

- If $0 < i < A(t_n)$, then there exists a decomposition (5) of $i$ such that $i_{j,l} = 0$ for every $j < n$ and there are $l_1, l_2$ such that $i_{n,l_1} = 1$ and $i_{n,l_2} = 0$. For this decomposition, the expression (4) is equal to $f(t_n) \wedge g(t_n)$. Besides, any decomposition (5) of $i$ will have some $i_{n,l_2} = 0$, and hence the value (4) for it will be $f(t_j) \wedge g(t_n)$. Since $f$ is increasing, $f(t_n) \wedge g(t_n)$ will be the maximum of all these outcomes, and hence the value of $S(A)(i)$.

- Finally, if $i = 0$, then the only decomposition (5) of $i$ is the one with all summands 0. For this decomposition, the expression (4) is equal to $g(t_n)$, and this will be the value of $S(A)(0)$.

This finishes the proof of the equality $\mathcal{C}(A) = S(A)$ when $f(0) = 0$.

**(I.b) $\mathbf{f(0) = 1}$.** If $f(0) = 1$, then $f$ is the constant mapping 1 and therefore $\mathcal{C}(A)(i) = g([A]_{i+1})$ for every $A \in FMS(]0,1])$ and $i \in \mathbb{N}$. If $Supp(A) = \{t_1, \ldots, t_n\} \neq \emptyset$, $t_1 < \cdots < t_n$, then, from the description of $f([A]_i) \wedge g([A]_{i+1})$ given above and using that $f(t_i) = 1$ for every $i = 1, \ldots, n$, we obtain that

$$g([A]_{i+1}) = \begin{cases} g(t_n) & \text{if } 0 < i+1 \leqslant A(t_n), \text{ i.e., if } 0 \leqslant i < A(t_n) \\ g(t_{n-1}) & \text{if } A(t_n) < i+1 \leqslant A(t_n) + A(t_{n-1}), \text{ i.e.,} \\ & \quad \text{if } A(t_n) \leqslant i < A(t_n) + A(t_{n-1}) \\ \vdots & \\ g(t_s) & \text{if } \sum_{j=s+1}^{n} A(t_j) < i+1 \leqslant \sum_{j=s}^{n} A(t_j), \text{ i.e.,} \\ & \quad \text{if } \sum_{j=s+1}^{n} A(t_j) \leqslant i < \sum_{j=s}^{n} A(t_j) \\ \vdots & \\ g(t_1) & \text{if } \sum_{j=2}^{n} A(t_j) < i+1 \leqslant \sum_{j=1}^{n} A(t_j), i.e., \\ & \quad \text{if } \sum_{j=2}^{n} A(t_j) \leqslant i < \sum_{j=1}^{n} A(t_j) \\ 1 & \text{if } \sum_{j=1}^{n} A(t_j) < i+1, \text{ i.e., if } \sum_{j=1}^{n} A(t_j) \leqslant i \end{cases}$$

As before, $S(A)(i)$ is the greatest value of expression (4) above for decompositions (5) of $i$. In this expression, and since $f$ is the constant mapping 1, every $\mathcal{C}(1/t_j)(i_{j,l})$ with $i_{j,l} \geqslant 1$ is 1: if $i_{j,l} = 1$, it is $f(t_j) = 1$ and if $i_{j,l} \geqslant 2$, it is $f(0) = 1$. Therefore, when we compute (4), all these 1's disappear and this expression is either equal to 1 (if every $i_{j,l} > 0$ in it) or to some

$$\mathcal{C}(1/t_{j_1})(0) \wedge \cdots \wedge \mathcal{C}(1/t_{j_k})(0) = g(t_{j_1}) \wedge \cdots \wedge g(t_{j_k}) = g(t_{j_k})$$

for some $j_1, \ldots, j_k \in \{1, \ldots, n\}$ such that $t_{j_1} < \cdots < t_{j_k}$ (these are exactly the indexes $j$ such that $i_{j,l} = 0$ for some $l$); in the last equality we have used that $g$ is decreasing.

Let us check now that $S(A)(i) = g([A]_{i+1})$ for every $i \in \mathbb{N}$ by dividing $\mathbb{N}$ into the same intervals as in the explicit description of the values $g([A]_{i+1})$ given above.

- If $i \geqslant \sum_{j=1}^{n} A(t_j)$, then there exists a decomposition of $i$ as in (5) with $i_{j,l} > 0$ for every $j = 1, \ldots, n$ and $l = 1, \ldots, A(t_j)$, which entails that $\mathcal{C}(A)(i) = 1$.

- If $\sum_{j=2}^{n} A(t_j) \leqslant i < \sum_{j=1}^{n} A(t_j)$, then there exists a decomposition (5) of $i$ with $i_{j,l} = 1$ for every $j > 1$ and for every $l = 1, \ldots, A(t_j)$, and some $i_{1,l} = 0$. The expression (4) corresponding to this decomposition is equal to $g(t_1)$. And for any other decomposition of $i$ this expression is equal to some $g(t_j)$ with $j \geqslant 1$ (because every decomposition of $i$ uses some 0). Since $g$ is decreasing, the maximum of these outcomes, and hence $\mathcal{C}(A)(i)$, is $g(t_1)$.

- In general, for every $s = 1, \ldots, n-1$, if

$$\sum_{j=s+1}^{n} A(t_j) \leqslant i < \sum_{j=s}^{n} A(t_j),$$

there exists a decomposition of $i$ as in (5) with $i_{j,l} = 1$ for every $j > s$ and for every $l = 1, \ldots, A(t_j)$ and some $i_{s,l} = 0$. The expression (4) corresponding to this decomposition is equal to $g(t_s)$. And this expression is equal to some $g(t_j)$ with $j \geqslant s$ for any other decomposition of $i$ (because there cannot exist any decomposition of $i$ with less or equal than $A(t_1) + \cdots + A(t_{s-1})$ 0's). Since $g$ is decreasing, the greatest of these results is $g(t_s)$, and hence $\mathcal{C}(A)(i) = g(t_s)$.

- Finally, if

$$0 \leqslant i < A(t_n),$$

every decomposition of $i$ as in (5) must have some $i_{n,l} = 0$. Therefore, every expression (4) in this case is equal to $g(t_n)$ and hence $\mathcal{C}(A)(i) = g(t_n)$.

This finishes the proof in the case $f(0) = 1$, and with it the proof of the alternative description of $\mathcal{C}_{f,g}$ given in this point.

Notice in particular that, since $f$ and $g$ are determined by the values of $\mathcal{C}_{f,g}$ on the singletons $1/t$, this description implies that if $\mathcal{C}_{f,g} = \mathcal{C}_{f',g'}$, then $f = f'$ and $g = g'$, thus proving the second part of the statement.

**(II) Every $\mathcal{C}_{f,g}$ is a fuzzy cardinality.** We must check that $\mathcal{C}_{f,g}$ satisfies conditions (i) to (iv) in Definition 2. As in the previous point, and to simplify the notations, once fixed the mappings $f$ and $g$, we shall denote $\mathcal{C}_{f,g}$ by $\mathcal{C}$.

(i) (**Additivity**) Let $A, B \in FMS(]0,1])$.

Assume first that one of them, say $B$, is the null multiset $\perp$. We must prove that $\mathcal{C}(A) = \mathcal{C}(A) \oplus \mathcal{C}(\perp)$. To do it, we distinguish two cases.

If $f(0) = 0$, then, as we saw in (I),

$$\mathcal{C}(\perp)(0) = 1 \text{ and } \mathcal{C}(\perp)(i) = 0 \text{ for every } i \geqslant 1;$$

in other words, $\mathcal{C}(\perp) = \overline{0}$, the neutral element of the generalized sum, and added (in the generalized sense) to any generalized natural number $\nu$, and not only those of the form $\mathcal{C}(A)$, yields $\nu$ again.

If $f(0) = 1$, i.e., if $f$ is the constant mapping 1, then, as we also saw in (I), $\mathcal{C}(\perp)(i) = 1$ for every $i \geqslant 0$. Notice moreover that, in this case, each $\mathcal{C}(A)$ is increasing (see Proposition 10 in the main body of the paper). Then, for every $A \in FMS(]0,1])$ and for every $i \in \mathbb{N}$,

$$\begin{aligned}(\mathcal{C}(A) \oplus \mathcal{C}(\perp))(i) &= \bigvee\{\mathcal{C}(A)(j) \wedge \mathcal{C}(\perp)(i-j) \mid j = 0, \dots, i\} \\ &= \bigvee\{\mathcal{C}(A)(j) \wedge 1 \mid j = 0, \dots, i\} \\ &= \bigvee\{\mathcal{C}(A)(0), \dots, \mathcal{C}(A)(i)\} = \mathcal{C}(A)(i).\end{aligned}$$

Assume now that $A$ and $B$ are both non-null. Then, the descriptions of $\mathcal{C}(A)$ and $\mathcal{C}(B)$ given in (I), together with the associativity and the commutativity of the extended sum in $\overline{\mathbb{N}}$, imply that

$$\begin{aligned}\mathcal{C}(A+B) &= \bigoplus_{t \in Supp(A+B)} \overbrace{\mathcal{C}(1/t) \oplus \cdots \oplus \mathcal{C}(1/t)}^{A(t)+B(t)} \\ &= \Big(\bigoplus_{t \in Supp(A+B)} \overbrace{\mathcal{C}(1/t) \oplus \cdots \oplus \mathcal{C}(1/t)}^{A(t)}\Big) \oplus \Big(\bigoplus_{t \in Supp(A+B)} \overbrace{\mathcal{C}(1/t) \oplus \cdots \oplus \mathcal{C}(1/t)}^{B(t)}\Big) \\ &= \Big(\bigoplus_{t \in Supp(A)} \overbrace{\mathcal{C}(1/t) \oplus \cdots \oplus \mathcal{C}(1/t)}^{A(t)}\Big) \oplus \Big(\bigoplus_{t \in Supp(B)} \overbrace{\mathcal{C}(1/t) \oplus \cdots \oplus \mathcal{C}(1/t)}^{B(t)}\Big) \\ &= \mathcal{C}(A) \oplus \mathcal{C}(B)\end{aligned}$$

(ii) (**Variability**) Recall from (I) that if if $i > Sc_1(A)$, then $[A]_i = 0$. Then, for every $i \in Sc_1(A)$,

$$\mathcal{C}(A)(i) = f([A]_i) \wedge g([A]_{i+1}) = f(0) \wedge g(0) = f(0),$$

which does not depend on $A$, and belongs to $\{0, 1\}$.

(iii) (**Consistency**) If $A = n/1$ with $n > 0$, then $[A]_i = 1$ for every $i = 0, \dots, n$ and $[A]_i = 0$ for $i \geqslant n+1$. Therefore

$$\mathcal{C}(A)(i) = f([A]_i) \wedge g([A]_{i+1}) = \begin{cases} f(1) \wedge g(1) = g(1) & \text{if } i \leqslant n-1 \\ f(1) \wedge g(0) = 1 & \text{if } i = n \\ f(0) \wedge g(0) = f(0) & \text{if } i \geqslant n+1 \end{cases}$$

(iv) (**Monotonicity**) We know from (I) that, for every $t \in ]0,1]$,

$$\mathcal{C}_{f,g}(1/t)(0) = g(t), \quad \mathcal{C}_{f,g}(1/t)(1) = f(t).$$

Then (iv) holds $g$ is decreasing and $f$ is increasing.

**(III) Every fuzzy cardinality is of the form $\mathcal{C}_{f,g}$.** Let $\mathcal{C} : FMS(]0,1]) \to \overline{\mathbb{N}}$ be a fuzzy cardinality. Consider the mappings $f, g : [0,1] \to [0,1]$ defined, for every $t \in ]0,1]$, by

$$f(t) = \mathcal{C}(1/t)(1), \quad g(t) = \mathcal{C}(1/t)(0),$$

and let $f(0) = \mathcal{C}(\perp)(1)$ and $g(0) = 1$.

Let us prove that these functions satisfy the properties required in the statement.

- $f$ is increasing by the monotonicity of fuzzy cardinalities.
- $g$ is decreasing on $]0,1]$ also by the monotonicity property, and since $g(0) = 1$, it is clear that it is decreasing on the whole interval $[0,1]$.
- The consistency of fuzzy cardinalities implies that $g(1) = \mathcal{C}(1/1)(0) \in \{0,1\}$, $f(1) = \mathcal{C}(1/1)(1) = 1$ and $f(0) = \mathcal{C}(0/1)(1) \in \{0,1\}$.

Finally, let us prove that $\mathcal{C} = \mathcal{C}_{f,g}$. It is clear that $\mathcal{C}(\perp) = \mathcal{C}_{f,g}(\perp)$. Moreover, $\mathcal{C}(1/t) = \mathcal{C}_{f,g}(1/t)$ for every $t \in ]0,1]$, because

$$\mathcal{C}(1/t)(0) = g(t) = \mathcal{C}_{f,g}(1/t)(0),\ \mathcal{C}(1/t)(1) = f(t) = \mathcal{C}_{f,g}(1/t)(1),$$
$$\mathcal{C}(1/t)(i) = \mathcal{C}(\perp)(1) = f(0) = \mathcal{C}_{f,g}(1/t)(i) \text{ for every } i \geqslant 2$$

(the equality $\mathcal{C}(1/t)(i) = \mathcal{C}(\perp)(1)$ is a consequence of the variability property). And then, the additivity of fuzzy cardinalities entails that, for every $A \in FMS(]0,1]) - \{\perp\}$,

$$\begin{aligned}\mathcal{C}(A) &= \bigoplus_{t \in Supp(A)} \overbrace{\mathcal{C}(1/t) \oplus \cdots \oplus \mathcal{C}(1/t)}^{A(t)} \\ &= \bigoplus_{t \in Supp(A)} \overbrace{\mathcal{C}_{f,g}(1/t) \oplus \cdots \oplus \mathcal{C}_{f,g}(1/t)}^{A(t)} = \mathcal{C}_{f,g}(A).\end{aligned}$$

This finishes the proof of Theorem 9.