The weak hereditary class of a variety

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Abstract

We study the *weak hereditary class* $S_w(\mathcal{K})$ of all weak subalgebras of algebras in a total variety $\mathcal{K}$. We establish an algebraic characterization, in the sense of Birkhoff's HSP theorem, and a syntactical characterization of these classes. We also consider the problem of when such a weak hereditary class is weak equational.

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The usefulness of partial algebras derives mostly from their extensibility to total algebras belonging to some specific variety or satisfying some other specific conditions. This feature has been frequently used in the literature, just to mention the original proof of the Grätzer-Schmidt theorem on the characterization of congruence lattices [3]. In this paper we intend to study the class $S_w(\mathcal{K})$ of all partial algebras embeddable in a variety of total algebras $\mathcal{K}$, and we call this class the *weak hereditary class* of $\mathcal{K}$. Thus, $S_w(\mathcal{K})$ can be seen as the class of all partial structural approximations of algebras in $\mathcal{K}$; the notation reflects the obvious fact that a partial algebra is embeddable in $\mathcal{K}$ if and only if it is (up to isomorphism) a weak subalgebra of an algebra in $\mathcal{K}$.

Clearly any algebra $\mathbf{A}$ in $S_w(\mathcal{K})$ must have the entire equational theory $\mathrm{Eq}(\mathcal{K})$ among the equations that are weakly valid in $\mathbf{A}$, i.e., $S_w(\mathcal{K}) \subseteq \mathrm{Mod}_w(\mathrm{Eq}(\mathcal{K}))$. As we shall see several times in this paper, this condition is by no means sufficient, and the converse inclusion is not generally true. It may be interesting however to distinguish those varieties of total algebras $\mathcal{K}$ for which $S_w(\mathcal{K})$ is a weak equational class, be it for the simple reason that weak equational theories are decidable [8]. In this paper we prove some general conditions for this to hold true, and in a subsequent paper we plan to investigate exhaustively this question for unary varieties.

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Notations and conventions. Once a type of algebras Σ is fixed, by a partial algebra (resp. an equation) we mean a partial algebra (resp. an equation) of type Σ. In this paper we only consider finitary homogeneous types.

Given a partial algebra denoted by a capital letter in boldface type ($\mathbf{A}$, $\mathbf{B}$, etc.), we shall always denote its carrier by the same capital letter in slanted type (A, B, etc.) and any operation in it by superscripting the operation symbol with the algebra's name ($\varphi^{\mathbf{A}}, \psi^{\mathbf{B}}, \ldots$).

Given a set $\mathcal{E}$ of Σ-equations, we denote by $\mathrm{Mod}_t(\mathcal{E})$ the *total variety* defined by $\mathcal{E}$, i.e., the class of all total Σ-algebras that satisfy $\mathcal{E}$. In a similar way, we denote by $\mathrm{Mod}_w(\mathcal{E})$ the *weak equational class* defined by $\mathcal{E}$, i.e., the class of all partial Σ-algebras that satisfy $\mathcal{E}$ weakly.

Given a total variety $\mathcal{K}$, $S_w(\mathcal{K})$ is the class of all weak subalgebras of algebras in $\mathcal{K}$.

All terms are supposed to have their variables in a fixed, countably infinite set of variables $\mathcal{X} = \{x_i \mid i \geq 1\}$. Given a term $\mathbf{p}$, we shall denote by $\mathrm{var}(\mathbf{p})$ its set of explicit variables.

1 Weak hereditary classes

In the sequel, and unless otherwise stated, let $\Sigma = (\Omega, \eta)$ be a fixed type of algebras.

Definition 1. A class $\mathcal{M}$ of partial algebras is *weak hereditary* if and only if $\mathcal{M} = S_w(\mathcal{K})$ for some total variety $\mathcal{K}$; in particular, we say that $S_w(\mathcal{K})$ is the *weak hereditary class* of $\mathcal{K}$.

Example 2. Let $\mathcal{E}$ be a set of equations and $\mathcal{K} = \mathrm{Mod}_t(\mathcal{E})$ the variety defined by $\mathcal{E}$. If a partial algebra $\mathbf{A}$ satisfies $\mathcal{E}$ existentially, then any *inner completion* of $\mathbf{A}$ (i.e., any completion of $\mathbf{A}$ on its carrier set) satisfies $\mathcal{E}$; thus, any existential model of $\mathcal{E}$ belongs to $S_w(\mathcal{K})$. (Actually, it is proved in [2] that any existential model of $\mathcal{E}$ is a *relative* subalgebra of a total model of $\mathcal{E}$.)

Example 3. Let now $\mathcal{E}$ be a set of *regular* equations and $\mathcal{K} = \mathrm{Mod}_t(\mathcal{E})$. If a partial algebra $\mathbf{A}$ satisfies $\mathcal{E}$ in the strong (or Kleene's) sense, then the one-point completion of $\mathbf{A}$ satisfies $\mathcal{E}$ (see [7]). Therefore, any strong model of such a set $\mathcal{E}$ belongs to $S_w(\mathcal{K})$.

More in general, if $\mathcal{E}$ is any set of equations closed under Birkhoff's equational logic rules, then any strong model of $\mathcal{E}$ belongs to $S_w(\mathcal{K})$: see Corollary 12 below. Thus, if $\mathbf{A}$ satisfies strongly the *whole* equational theory $\mathrm{Eq}(\mathcal{K})$ of a total variety $\mathcal{K}$, then $\mathbf{A} \in S_w(\mathcal{K})$. However, when $\mathcal{E}$ is not closed in the Birkhoff sense, strong models of $\mathcal{E}$ need not be weak subalgebras of total models of $\mathcal{E}$, as the following simple counterexample shows.

Example 4. Let Σ be a type of unary algebras with at least three different operation symbols φ_0, φ_1 and φ_2, and let $\mathbf{A}$ be a partial algebra with carrier $A = \{a, b\}$ and with $\varphi_0^{\mathbf{A}}(a) = \varphi_0^{\mathbf{A}}(b) = a$, $\varphi_1^{\mathbf{A}}(a) = \varphi_1^{\mathbf{A}}(b) = b$, and $\varphi_2^{\mathbf{A}}$ discrete. Then $\mathbf{A}$ is a strong model of the equations

$$\varphi_0(x_1) \approx \varphi_0(x_2),\ \varphi_1(x_1) \approx \varphi_1(x_2),\ \varphi_0\varphi_2(x_1) \approx \varphi_1\varphi_2(x_2)$$

but no completion of $\mathbf{A}$ can satisfy them. Indeed, any total model of these equations also satisfies $\varphi_0(x_1) \approx \varphi_1(x_2)$, while $\varphi_0^{\mathbf{A}}(a) \neq \varphi_1^{\mathbf{A}}(b)$.

Last example entails in particular that, although any algebra in $S_w(\mathrm{Mod}_t(\mathcal{E}))$ is a weak model of $\mathcal{E}$ because weak subalgebras preserve weak satisfaction of equations, the converse implication is in general false. Let us see another, simpler, example of this fact.

Example 5. Let Σ be a type of monounary algebras, and let φ be its operation symbol. Let $\mathbf{A}$ be a partial algebra with carrier $A = \{a, a', a''\}$ and with operation $\varphi^{\mathbf{A}}$ given by $\varphi^{\mathbf{A}}(a) = \varphi^{\mathbf{A}}(a') = a''$. Then $\mathbf{A}$ satisfies $\varphi^2(x) \approx x$ in the weak sense, but clearly no completion of $\mathbf{A}$ satisfies this equation.

Thus, a weak model of the whole equational theory of a total variety need not belong to its weak hereditary class. Indeed, this algebra $\mathbf{A}$ satisfies weakly the whole equational theory of $\mathrm{Mod}_t(\{\varphi^2(x) \approx x\})$, but it does not belong to $S_w(\{\varphi^2(x) \approx x\})$.

Remark 6. A weak hereditary class is closed under isomorphisms, weak subalgebras and arbitrary products. Hence, it is quasi-equational: actually, by [1, §3, Th. 1], $\mathbf{A} \in S_w(\mathcal{K})$ if and only if

$$\mathbf{A} \models \bigwedge_{j \in J} \mathbf{p}_j \approx \mathbf{q}_j \rightarrow x \approx y$$

for every such quasi-equation holding in $\mathcal{K}$. In Theorem 13 below we give another syntactical characterization of $S_w(\mathcal{K})$.

Next result provides an algebraic characterization of weak hereditary classes in the sense of Birkhoff's HSP Theorem. In it, and in the sequel, H, S, S_t and P stand for the "closed homomorphic images", "closed subalgebras", "total (closed) subalgebras" and "direct products" algebraic operators, respectively. Moreover, given a class $\mathcal{M}$ of partial algebras, $\mathcal{M}^*$ denotes the subclass of all total algebras in $\mathcal{M}$. Notice that $S_w(\mathcal{K})^* = \mathcal{K}$ for every total variety $\mathcal{K}$.

Theorem 7. *Let $\mathcal{M}$ be a class of partial algebras. The following assertions on $\mathcal{M}$ are equivalent:*

i) $\mathcal{M}$ is a weak hereditary class.

ii) $\mathcal{M} = S_w HSP(\mathcal{M}^*)$.

iii) $\mathcal{M} = S_w HS_t P(\mathcal{M})$.

Proof. $(i) \Longrightarrow (ii)$ If $\mathcal{M} = S_w(\mathcal{K})$ for some variety $\mathcal{K}$ of total algebras, then $\mathcal{M}^* = \mathcal{K} = HSP(\mathcal{K}) = HSP(\mathcal{M}^*)$ and therefore $\mathcal{M} = S_w(HSP(\mathcal{M}^*))$.

$(ii) \Longrightarrow (iii)$ Assume $\mathcal{M} = S_w HSP(\mathcal{M}^*)$. Then

$$P(\mathcal{M}) = PS_w HSP(\mathcal{M}^*) \subseteq S_w PHSP(\mathcal{M}^*) \subseteq S_w HSP(\mathcal{M}^*) = \mathcal{M}$$

and

$$S_t(\mathcal{M}) = S_t S_w HSP(\mathcal{M}^*) \subseteq S_w HSP(\mathcal{M}^*) = \mathcal{M}.$$

Therefore $S_t P(\mathcal{M}) \subseteq \mathcal{M}$, and, in fact, $S_t P(\mathcal{M}) \subseteq \mathcal{M}^*$.

Consequently,

$$S_w H S_t P(\mathcal{M}) \subseteq S_w H(\mathcal{M}^*) \subseteq S_w HSP(\mathcal{M}^*) = \mathcal{M}.$$

On the other hand, the inclusion $\mathcal{M} \subseteq S_w H S_t P(\mathcal{M})$ follows from the fact that $SP(\mathcal{M}^*) \subseteq S_t P(\mathcal{M})$ for any class $\mathcal{M}$ (if $\mathbf{A}$ is a closed subalgebra of a total algebra, $\mathbf{A}$ is total too), and therefore $\mathcal{M} = S_w HSP(\mathcal{M}^*) \subseteq S_w H S_t P(\mathcal{M})$.

$(iii) \Longrightarrow (i)$ Assume now that $\mathcal{M} = S_w H S_t P(\mathcal{M})$. We shall prove that, under this assumption, $HS_t P(\mathcal{M}) = HSP(\mathcal{M}^*)$. Indeed, on the one hand, the inclusion $SP(\mathcal{M}^*) \subseteq S_t P(\mathcal{M})$ used in the last step of the proof of the previous implication entails that $HSP(\mathcal{M}^*) \subseteq HS_t P(\mathcal{M})$ for any class $\mathcal{M}$. And on the other hand, if $\mathbf{B} \in HS_t P(\mathcal{M})$ then $\mathbf{B}$ is total and belongs to $\mathcal{M}$ because $\mathcal{M} = S_w H S_t P(\mathcal{M})$, and therefore $\mathbf{B} \in \mathcal{M}^* \subseteq HSP(\mathcal{M}^*)$; this proves $HS_t P(\mathcal{M}) \subseteq HSP(\mathcal{M}^*)$.

Thus, $HS_t P(\mathcal{M})$ is a variety of total algebras and $\mathcal{M}$ is its weak hereditary class. □

Remark 8. Notice that the equality $HS_t P(\mathcal{M}) = HSP(\mathcal{M}^*)$ need not hold for an arbitrary class $\mathcal{M}$ of partial algebras. For instance, let $\mathbf{A}$ be a non-total algebra containing a total and non-trivial closed subalgebra $\mathbf{A}'$, and let $\mathcal{M} = \{\mathbf{A}\}$. Then $\mathbf{A}' \in HS_t P(\mathcal{M})$, while $HSP(\mathcal{M}^*)$ only contains trivial total algebras.

In the sequel we provide an alternative syntactical characterization of weak hereditary classes. Recall that given a class $\mathcal{K}$ of similar algebras and an algebra $\mathbf{A}$ of the same type, a *$\mathcal{K}$-reflection* of $\mathbf{A}$ is, when it exists, a homomorphism $e : \mathbf{A} \to \mathbf{B}$ with $\mathbf{B} \in \mathcal{K}$ such that every homomorphism of $\mathbf{A}$ to an algebra in $\mathcal{K}$ factorizes in a unique way through e. Since such an algebra $\mathbf{B}$ is determined uniquely (up to isomorphism) by $\mathbf{A}$, we shall call it *the $\mathcal{K}$-reflection* of $\mathbf{A}$ whenever e is understood or nonrelevant. The first lemma is an easy consequence of the universal property of $\mathcal{K}$-reflections.

Lemma 9. *Let $\mathcal{K}$ be a total variety and $\mathbf{A}$ a partial algebra. Then, $\mathbf{A} \in S_w(\mathcal{K})$ if and only if $\mathbf{A}$ is (up to isomorphism) a weak subalgebra of its $\mathcal{K}$-reflection.* □

Next lemma describes in a simple (and suitable for our purposes) way the reflection of a partial algebra with respect to a variety of total algebras.

Lemma 10. *Let $\mathbf{A}$ be a partial algebra and $i_{\mathbf{A}} : \mathbf{A} \hookrightarrow \mathbf{F}$ a free completion of it (given by the embedding of $\mathbf{A}$ into $\mathbf{F}$ as a relative subalgebra), and let $\mathcal{K}$ be a total variety.*

Let $\theta(\mathcal{K})$ be the transitive closure in $\mathbf{F}$ *of the relation*

$$\varepsilon(\mathcal{K}) = \{(\mathbf{p}^{\mathbf{F}}(v), \mathbf{q}^{\mathbf{F}}(v)) \mid \mathbf{p} \approx \mathbf{q} \in \mathrm{Eq}(\mathcal{K}),\ v \in A^{\mathcal{X}}\}.$$

Then $\theta(\mathcal{K})$ is a congruence on $\mathbf{F}$ *such that $\mathbf{F}/_{\theta(\mathcal{K})} \in \mathcal{K}$, and the composition*

$$\mathrm{nat}_{\,\theta(\mathcal{K})} \circ i_{\mathbf{A}} : \mathbf{A} \to \mathbf{F}/_{\theta(\mathcal{K})}$$

is the $\mathcal{K}$-reflection of $\mathbf{A}$.

Proof. It is clear that $\theta(\mathcal{K})$ is an equivalence relation: it is reflexive because $\mathbf{F}$ is generated by A, it is symmetric because $\mathrm{Eq}(\mathcal{K})$ is symmetric, and it is transitive by construction. Now, let $\varphi \in \Omega$ with $\eta(\varphi) = n \geq 1$, and let $(a_1, b_1), \ldots, (a_n, b_n) \in \theta(\mathcal{K})$. Without any loss of generality, we may assume that there exist an integer $m \geq 1$ and pairs $(a_{i,j}, a_{i,j+1}) \in \varepsilon(\mathcal{K})$, $i = 1, \ldots, n$, $j = 0, \ldots, m-1$, such that $a_{i,0} = a_i$ and $a_{i,m} = b_i$.

Then, for every $j = 0, \ldots, m-1$, there exist equations $\mathbf{p}_{i,j} \approx \mathbf{q}_{i,j} \in \mathrm{Eq}(\mathcal{K})$ and valuations $v_{i,j} : \mathcal{X} \to A$ such that $a_{i,j} = \mathbf{p}_{i,j}^{\mathbf{F}}(v_{i,j})$ and $a_{i,j+1} = \mathbf{q}_{i,j}^{\mathbf{F}}(v_{i,j})$, $i = 1, \ldots, n$; $\mathrm{Eq}(\mathcal{K})$ being closed under substitution, we may assume that all these equations have pairwise disjoint sets of explicit variables. But then, since $\mathrm{Eq}(\mathcal{K})$ is closed under composition,

$$\varphi(\mathbf{p}_{1,j}, \ldots, \mathbf{p}_{n,j}) \approx \varphi(\mathbf{q}_{1,j}, \ldots, \mathbf{q}_{n,j}) \in \mathrm{Eq}(\mathcal{K}).$$

Therefore, taking a valuation $v : \mathcal{X} \to A$ such that its restriction to every $\mathrm{var}\,(\mathbf{p}_{i,j}) \cup \mathrm{var}\,(\mathbf{q}_{i,j})$ is equal to the corresponding restriction of $v_{i,j}$, we have that

$$\begin{aligned}(\varphi^{\mathbf{F}}(a_{1,j}, \ldots, a_{n,j}), \varphi^{\mathbf{F}}(a_{1,j+1}, \ldots, a_{n,j+1}))& \\ = (\varphi(\mathbf{p}_{1,j}, \ldots, \mathbf{p}_{n,j})^{\mathbf{F}}(v), \varphi(\mathbf{q}_{1,j}, \ldots, \mathbf{q}_{n,j})^{\mathbf{F}}(v)) &\in \varepsilon(\mathcal{K})\end{aligned}$$

for every $j = 0, \ldots, m-1$. Then, finally,

$$(\varphi^{\mathbf{F}}(a_1, \ldots, a_n), \varphi^{\mathbf{F}}(b_1, \ldots, b_m)) = (\varphi^{\mathbf{F}}(a_{1,0}, \ldots, a_{n,0}), \varphi^{\mathbf{F}}(a_{1,m}, \ldots, a_{n,m})) \in \theta(\mathcal{K}).$$

Thus, $\theta(\mathcal{K})$ is indeed a congruence on $\mathbf{F}$.

We show now that $\mathbf{F}/_{\theta(\mathcal{K})}$ satisfies $\mathrm{Eq}(\mathcal{K})$. Let $\mathbf{p} \approx \mathbf{q} \in \mathrm{Eq}(\mathcal{K})$, say with $\mathrm{var}\,(\mathbf{p}) \cup \mathrm{var}\,(\mathbf{q}) = \{x_1, \ldots, x_n\}$, and let $([c_1]_\theta, \ldots, [c_n]_\theta)$ be any n-tuple of elements of $F/_{\theta(\mathcal{K})}$.

Since $\mathbf{F}$ is generated by A, for every c_i there exists a term $\mathbf{p}_i$ and a valuation $v_i : \mathcal{X} \to A$ such that $c_i = \mathbf{p}_i^{\mathbf{F}}(v_i)$; as before, we may assume that the sets of variables of these terms are pairwise disjoint. Then

$$\mathbf{p}(x_1/\mathbf{p}_1, \ldots, x_n/\mathbf{p}_n) \approx \mathbf{q}(x_1/\mathbf{p}_1, \ldots, x_n/\mathbf{p}_n) \in \mathrm{Eq}(\mathcal{K}).$$

But now, taking a valuation $v : \mathcal{X} \to A$ such that its restriction to every $\mathrm{var}\,(\mathbf{p}_i)$ is equal to the corresponding restriction of v_i, we have that

$$(\mathbf{p}^{\mathbf{F}}(c_1, \ldots, c_n), \mathbf{q}^{\mathbf{F}}(c_1, \ldots, c_n)) \\ = (\mathbf{p}(x_1/\mathbf{p}_1, \ldots, x_n/\mathbf{p}_n)^{\mathbf{F}}(v), \mathbf{q}(x_1/\mathbf{p}_1, \ldots, x_n/\mathbf{p}_n)^{\mathbf{F}}(v)) \in \varepsilon(\mathcal{K}),$$

and therefore

$$\mathbf{p}^{\mathbf{F}/\theta}([c_1]_\theta, \ldots, [c_n]_\theta) = \mathbf{q}^{\mathbf{F}/\theta}([c_1]_\theta, \ldots, [c_n]_\theta).$$

This implies that $\mathbf{F}/_{\theta(\mathcal{K})}$ satisfies $\mathbf{p} \approx \mathbf{q}$.

Finally, let $f : \mathbf{F} \to \mathbf{B}$ be any homomorphism with $\mathbf{B} \in \mathcal{K}$. Then obviously $\theta(\mathcal{K}) \subseteq \ker f$, and therefore there exists a unique homomorphism $\tilde{f} : \mathbf{F}/_{\theta(\mathcal{K})} \to \mathbf{B}$ such that $f = \tilde{f} \circ \mathrm{nat}_{\theta(\mathcal{K})}$. This means that $\mathrm{nat}_{\theta(\mathcal{K})} : \mathbf{F} \to \mathbf{F}/_{\theta(\mathcal{K})}$ is a $\mathcal{K}$-reflection of $\mathbf{F}$. Since $\mathbf{F}$ is a free completion of $\mathbf{A}$, $\mathrm{nat}_{\theta(\mathcal{K})} \circ i_{\mathbf{A}} : \mathbf{A} \to \mathbf{F}/_{\theta(\mathcal{K})}$ is a $\mathcal{K}$-reflection of $\mathbf{A}$. □

The next two corollaries are simple consequences of the previous two lemmas.

Corollary 11. *With the notation and assumptions of Lemma 10,* $\mathbf{A} \in S_w(\mathcal{K})$ *if and only if* $\theta(\mathcal{K}) \cap A^2 = \Delta_A$ *(the diagonal on* A*).* □

Corollary 12. *If* $\mathcal{K}$ *is a total variety and* $\mathbf{A}$ *satisfies* $\mathrm{Eq}(\mathcal{K})$ *strongly, then* $\mathbf{A} \in S_w(\mathcal{K})$.

Proof. If $\mathbf{A}$ is a strong model of $\mathrm{Eq}(\mathcal{K})$, then, for every $(a, b) \in \varepsilon(\mathcal{K})$, $b = a$ whenever $a \in A$. This clearly implies that $\theta(\mathcal{K}) \cap A^2 = \Delta_A$, and the last corollary applies. □

In order to simplify the notation in the statement of the next result, we introduce the concept of *relative prefixes* of two Σ-terms. A term $\mathbf{t} = \mathbf{t}(x_1, \ldots, x_n) \in \mathrm{T}_\Sigma(\mathcal{X})$ (with explicit variables $x_1, \ldots, x_n$) is a *common suffix* of two terms $\mathbf{p}, \mathbf{q} \in \mathrm{T}_\Sigma(\mathcal{X})$ when there exist terms $\mathbf{p}_1, \ldots, \mathbf{p}_n, \mathbf{q}_1, \ldots, \mathbf{q}_n$ such that

$$\mathbf{p} = \mathbf{t}(x_1/\mathbf{p}_1, \ldots, x_n/\mathbf{p}_n),\ \mathbf{q} = \mathbf{t}(x_1/\mathbf{q}_1, \ldots, x_n/\mathbf{q}_n).$$

In this case, $(\mathbf{p}_1, \mathbf{q}_1), \ldots, (\mathbf{p}_n, \mathbf{q}_n)$ are called the *pairs of relative prefixes* of the pair $(\mathbf{p}, \mathbf{q})$ with respect to $\mathbf{t}$.

Theorem 13. *Let* $\mathbf{A}$ *be a partial* Σ*-algebra and* $\mathcal{K}$ *a variety of total* Σ*-algebras. Then,* $\mathbf{A} \in S_w(\mathcal{K})$ *if and only if* $\mathbf{A}$ *satisfies all quasi-equations of the form*

$$(\mathbf{p}_0 \approx \mathbf{p}_0 \wedge \mathbf{q}_k \approx \mathbf{q}_k \wedge \bigwedge_{j=0}^{k-1} \bigwedge_{l=1}^{n_j} \bar{\mathbf{q}}_{j,l} \approx \bar{\mathbf{p}}_{j+1,l}) \to \mathbf{p}_0 \approx \mathbf{q}_k,$$

where, for some sequence of equations

$$\mathbf{p}_0 \approx \mathbf{q}_0, \mathbf{p}_1 \approx \mathbf{q}_1, \ldots, \mathbf{p}_k \approx \mathbf{q}_k \in \mathrm{Eq}(\mathcal{K}), \quad k \geq 0,$$

with pairwise disjoint sets of explicit variables (i.e., such that $(\mathrm{var}\,(\mathbf{p}_i) \cup \mathrm{var}\,(\mathbf{q}_i)) \cap (\mathrm{var}\,(\mathbf{p}_j) \cup \mathrm{var}\,(\mathbf{q}_j)) = \emptyset$ *for every* $i \neq j$*),*

$$(\bar{\mathbf{q}}_{j,1}, \bar{\mathbf{p}}_{j+1,1}), \ldots, (\bar{\mathbf{q}}_{j,n_j}, \bar{\mathbf{p}}_{j+1,n_j})$$

are the pairs of relative prefixes of $(\mathbf{q}_j, \mathbf{p}_{j+1})$ *with respect to some common suffix, for every* $j = 0, \ldots, k-1$.

Proof. The "only if" implication is straightforward, because every algebra in $\mathcal{K}$ satisfies all quasi-equations of the form given in the statement, and the satisfaction of such quasi-equations is preserved by weak subalgebras.

As for the "if" implication, by Corollary 11 it is enough to prove that if $\mathbf{A}$ satisfies all quasi-equations of the form given in the statement, then $\theta(\mathcal{K}) \cap A^2 = \Delta_A$, where $\theta(\mathcal{K})$ is the congruence on the free completion $\mathbf{F}$ of $\mathbf{A}$ defined in Lemma 10.

So, let $(a, a') \in \theta(\mathcal{K})$ with $a, a' \in A$. Then, there exist

$$(b_0, b_1), (b_1, b_2), \ldots, (b_k, b_{k+1}) \in \varepsilon(\mathcal{K}), \quad k \geq 0,$$

such that $a = b_0$ and $a' = b_{k+1}$; without any loss of generality we shall assume that no other b_i in this sequence of pairs belongs to A.

By the definition of $\varepsilon(\mathcal{K})$, there exist equations $\mathbf{p}_0 \approx \mathbf{q}_0, \mathbf{p}_1 \approx \mathbf{q}_1, \ldots, \mathbf{p}_k \approx \mathbf{q}_k$ in $\mathrm{Eq}(\mathcal{K})$ (and, as always, we shall assume their sets of explicit variables to be pairwise disjoint) and valuations $v_0, \ldots, v_k : \mathcal{X} \to A$ such that, for every $i = 0, \ldots, k$,

$$(b_i, b_{i+1}) = (\mathbf{p}_i^{\mathbf{F}}(v_i), \mathbf{q}_i^{\mathbf{F}}(v_i)).$$

Since $\mathbf{A}$ is an initial segment of $\mathbf{F}$, $b_0, b_{k+1} \in A$ implies $b_0 = \mathbf{p}_0^{\mathbf{A}}(v_0)$ and $b_{k+1} = \mathbf{q}_k^{\mathbf{A}}(v_k)$. Moreover, from the other Peano axioms satisfied by the free completion $\mathbf{F}$ of $\mathbf{A}$ (see [2] or [6]), we deduce that if $\mathbf{q}_j^{\mathbf{F}}(v_j) = \mathbf{p}_{j+1}^{\mathbf{F}}(v_{j+1}) = b_{j+1} \notin A$, then there is a common suffix $\mathbf{t}_j$ of $\mathbf{q}_j$ and $\mathbf{p}_{j+1}$ such that, if

$$(\bar{\mathbf{q}}_{j,1}, \bar{\mathbf{p}}_{j+1,1}), \ldots, (\bar{\mathbf{q}}_{j,n_j}, \bar{\mathbf{p}}_{j+1,n_j})$$

are the pairs of relative prefixes of $(\mathbf{q}_j, \mathbf{p}_{j+1})$ with respect to $\mathbf{t}_j$, then $\bar{\mathbf{q}}_{j,l}^{\mathbf{A}}(v_j)$ and $\bar{\mathbf{p}}_{j+1,l}^{\mathbf{A}}(v_{j+1})$ are defined and equal for every $l = 1, \ldots, n_j$ (*grosso modo*, if $\mathbf{q}_j^{\mathbf{F}}(v_j) = \mathbf{p}_{j+1}^{\mathbf{F}}(v_{j+1}) \notin A$, then both must be equal to the same term, a common suffix $\mathbf{t}_j$ of $\mathbf{q}_j$ and $\mathbf{p}_{j+1}$, applied in $\mathbf{F}$ to the same elements of A, given by the application in $\mathbf{A}$ of the relative prefixes of $\mathbf{q}_j$ and $\mathbf{p}_{j+1}$ with respect to this common suffix to v_j and v_{j+1}, respectively).

Let now $v : \mathcal{X} \to A$ be a valuation such that its restriction to every $\text{var}(\mathbf{p}_i) \cup \text{var}(\mathbf{q}_i)$ is equal to the restriction of v_i to this set. We have that

$$(\mathbf{A}, v) \models \mathbf{p}_0 \approx \mathbf{p}_0 \wedge \mathbf{q}_k \approx \mathbf{q}_k \wedge \bigwedge_{j=0}^{k-1} \bigwedge_{l=1}^{n_j} \bar{\mathbf{q}}_{j,l} \approx \bar{\mathbf{p}}_{j+1,l}.$$

Then, by hypothesis,

$$(\mathbf{A}, v) \models \mathbf{p}_0 \approx \mathbf{q}_k$$

and therefore

$$a = \mathbf{p}_0^{\mathbf{A}}(v_0) = \mathbf{p}_0^{\mathbf{A}}(v) = \mathbf{q}_k^{\mathbf{A}}(v) = \mathbf{q}_k^{\mathbf{A}}(v_k) = a',$$

as we wanted to prove. □

2 When is $S_w(\mathcal{K})$ weak equational?

In this section we discuss some conditions for a class $S_w(\mathcal{K})$ of all "approximations" of algebras in a total variety $\mathcal{K}$ to be weakly equational. As we have seen in Section 1, all members of $S_w(\mathcal{K})$ are weak models of the equational theory $\text{Eq}(\mathcal{K})$ of $\mathcal{K}$. However, Examples 4 and 5 showed that, in general, not every weak model of $\text{Eq}(\mathcal{K})$ belongs to $S_w(\mathcal{K})$.

Lemma 14. *Let $\mathcal{K}$ be a variety of total algebras. Then* $\text{Mod}_w(\text{Eq}(\mathcal{K})) = HS_w(\mathcal{K})$.

Proof. By Andréka-Németi's general characterization theorem of model classes [1, §3, Th. 1], $\mathbf{A} \in HS_w(\mathcal{K})$ if and only if

$$\mathbf{A} \models (\mathbf{p} \approx \mathbf{p} \wedge \mathbf{q} \approx \mathbf{q} \wedge \bigwedge_{j \in J} \mathbf{p}_j \approx \mathbf{p}_j) \to \mathbf{p} \approx \mathbf{q}$$

for every such quasi-equation satisfied in $\mathcal{K}$, which in this case means, $\mathcal{K}$ being a total variety, simply that $\mathbf{p} \approx \mathbf{q} \in \text{Eq}(\mathcal{K})$. Since the weak satisfaction of an equation $\mathbf{p} \approx \mathbf{q}$ is nothing but the satisfaction of the quasi-equation

$$(\mathbf{p} \approx \mathbf{p} \wedge \mathbf{q} \approx \mathbf{q}) \to \mathbf{p} \approx \mathbf{q},$$

it is clear that $\mathbf{A} \in HS_w(\mathcal{K})$ if and only if $\mathbf{A}$ satisfies weakly all equations in $\text{Eq}(\mathcal{K})$. □

This result yields a general necessary and sufficient condition on $S_w(\mathcal{K})$ to be weak equational.

Proposition 15. *Let $\mathcal{K}$ be a variety of total algebras. The following conditions are equivalent.*

i) $S_w(\mathcal{K})$ *is a weak equational class.*

ii) $S_w(\mathcal{K}) = \mathrm{Mod}_w(\mathrm{Eq}(\mathcal{K}))$.

iii) $S_w(\mathcal{K})$ *is closed under closed homomorphic images.*

Proof. Notice that if $\mathcal{E}$ is the weak equational theory of $S_w(\mathcal{K})$ then $\mathcal{E} = \mathrm{Eq}\,(\mathcal{K})$ (since $\mathcal{K} \subseteq S_w(\mathcal{K})$, we have $\mathcal{E} \subseteq \mathrm{Eq}\,(\mathcal{K})$, and since weak satisfaction is preserved by weak subalgebras, $\mathrm{Eq}\,(\mathcal{K}) \subseteq \mathcal{E}$). Therefore, $S_w(\mathcal{K})$ is weak equational if and only if $S_w(\mathcal{K}) = \mathrm{Mod}_w(\mathrm{Eq}\,(\mathcal{K}))$, which proves the equivalence between (i) and (ii). The equivalence between (ii) and (iii) is now a consequence of Lemma 14. ☐

This proposition, together with Theorem 13, can be used to show that some specific total varieties do not have their weak hereditary classes weak equational. We end this paper with some examples.

Example 16. Let Σ be a monounary type of algebras with operation symbol φ, and let $\mathcal{K}$ be a total variety of Σ-algebras. Then $\mathrm{Eq}(\mathcal{K})$ is generated by a single equation: if $\mathcal{K}$ is regular, then $\mathcal{K} = \mathrm{Mod}_t(\varphi^r(x_1) \approx \varphi^s(x_1))$ for some $r \geq s \geq 0$, and if $\mathcal{K}$ is irregular, then $\mathcal{K} = \mathrm{Mod}_t(\varphi^r(x_1) \approx \varphi^r(x_2))$ for some $r \geq 0$; see [4].

In the regular case, i.e., when $\mathcal{K} = \mathrm{Mod}_t(\varphi^r(x_1) \approx \varphi^s(x_1))$, $S_w(\mathcal{K})$ is weak equational if and only if $r = s$ or $r = s + 1$. To prove this, we distinguish the following three cases:

- Assume first that $r > s + 1$. Consider the relative subalgebra $\mathbf{A}$ of the Σ-term algebra $\mathbf{T}_\Sigma(\{x, y\})$ supported on the initial segment determined by $\{\varphi^{r-1}(x), \varphi^{r-1}(y)\}$; it is easily seen to be a weak subalgebra of the $\mathcal{K}$-free algebra generated by $\{x, y\}$. Let θ denote the equivalence relation on A identifying $\varphi^{r-1}(x)$ and $\varphi^{r-1}(y)$. It is a closed congruence on $\mathbf{A}$, and thus $\mathbf{A}/_\theta \in HS_w(\mathcal{K})$. But $\mathbf{A}/_\theta \notin S_w(\mathcal{K})$, because it does not satisfy the quasi-equation

  $$\Big(\varphi^s(x_1) \approx \varphi^s(x_1) \wedge \varphi^s(x_2) \approx \varphi^s(x_2) \wedge \varphi^{r-1}(x_1) \approx \varphi^{r-1}(x_2)\Big) \\ \to \varphi^s(x_1) \approx \varphi^s(x_2),$$

  obtained from the equations $\varphi^s(x_1) \approx \varphi^r(x_1)$ and $\varphi^r(x_2) \approx \varphi^s(x_2)$ as in Theorem 13, with respect to any valuation $v : \mathcal{X} \to A/_\theta$ such that $v(x_1) = [x]_\theta$ and $v(x_2) = [y]_\theta$.

  So, if $r > s + 1$, $S_w(\mathcal{K})$ is not closed under closed homomorphic images, and therefore it is not weak equational.

- If $r = s$, then $\mathcal{K}$ is the class of all Σ-algebras, and therefore $S_w(\mathcal{K})$ is the (weak equational) class of all partial Σ-algebras.

- Finally, if $r = s + 1$, then completing any weak model $\mathbf{A}$ of $\varphi^{s+1}(x) \approx \varphi^s(x)$ by adding a φ-loop to every point where $\varphi^{\mathbf{A}}$ is not defined, yields a total Σ-algebra satisfying $\varphi^{s+1}(x_1) \approx \varphi^s(x_1)$, i.e. an algebra in $\mathcal{K}$. So, in this case every weak model of $\mathrm{Eq}(\mathcal{K})$ is in $S_w(\mathcal{K})$, which implies that the latter is a weak equational class.

In the irregular case, i.e., when $\mathcal{K} = \mathrm{Mod}_t(\varphi^r(x_1) \approx \varphi^r(x_2))$ for some $r \geq 0$, $S_w(\mathcal{K})$ is always a weak equational class, because any weak model $\mathbf{A}$ of $\varphi^r(x) \approx \varphi^r(y)$ can be extended to a total algebra $\overline{\mathbf{A}}$ satisfying it. To prove it, we have to distinguish three cases:

- If $r = 0$, then $\mathbf{A}$ is a trivial algebra and $\mathcal{K}$ is the class of all trivial total algebras, and then any inner completion of $\mathbf{A}$ belongs to $\mathcal{K}$.
- If $r > 0$ and $(\varphi^{\mathbf{A}})^r(a)$ is defined for some $a \in A$, then add $\varphi^{\overline{\mathbf{A}}}(a') = (\varphi^{\mathbf{A}})^r(a)$ for every $a' \notin \mathrm{dom}\,\varphi^{\mathbf{A}}$, and the resulting $\overline{\mathbf{A}}$ satisfies $\varphi^r(x_1) \approx \varphi^r(x_2)$.
- If $r > 0$ and $\mathrm{dom}\,(\varphi^{\mathbf{A}})^r = \emptyset$, then take any $a_0 \notin \mathrm{dom}\,\varphi^{\mathbf{A}}$ and add $\varphi^{\overline{\mathbf{A}}}(a') = a_0$ for every $a' \notin \mathrm{dom}\,\varphi^{\mathbf{A}}$ (including a_0 itself). Again, the resulting $\overline{\mathbf{A}}$ satisfies $\varphi^r(x_1) \approx \varphi^r(x_2)$.

In a subsequent paper we plan to characterize those of varieties of total algebras of an arbitrary unary type whose weak hereditary class is weak equational.

Example 17. Let Σ be a type of algebras with only two operation symbols $*$ and $+$, both binary, and let $\mathcal{K}$ be the total variety of Σ-algebras defined by the absorption equation $x_1 + (x_1 * x_2) \approx x_1$. The weak hereditary class $S_w(\mathcal{K})$ is weak equational. Indeed, for every partial Σ-algebra $\mathbf{A}$, let $\overline{\mathbf{A}}$ be its completion defined in the following way:

- $\overline{A} = A \cup \{a_0\}$ for some $a_0 \notin A$.
- For every $x, y \in \overline{A}$, if $x * y$ is not defined in $\mathbf{A}$, then $x * y = a_0$ in $\overline{\mathbf{A}}$.
- For every $x, y \in \overline{A}$, if $x + y$ is not defined in $\mathbf{A}$, then $x + y = x$ in $\overline{\mathbf{A}}$.

It is straightforward to check that if $\mathbf{A}$ satisfies weakly $x_1 + (x_1 * x_2) \approx x_1$, then $\overline{\mathbf{A}} \in \mathcal{K}$ and hence $\mathbf{A} \in S_w(\mathcal{K})$.

Let now $\mathcal{K}'$ be the total variety of Σ-algebras defined by *both* absorption equations

$$x_1 + (x_1 * x_2) \approx x_1,\ x_1 * (x_1 + x_2) \approx x_1.$$

Then, $S_w(\mathcal{K}')$ is not weak equational. Indeed, let $\mathbf{A}$ be a partial Σ-algebra with carrier $A = \{a, b\}$, operation $+$ totally defined by

$$a + a = a + b = b,\ b + a = b + b = a,$$

and operation $*$ discrete. It is clear then that $\mathbf{A}$ is a weak model of $\mathrm{Eq}(\mathcal{K}')$. But no completion of $\mathbf{A}$ belongs to $\mathcal{K}'$, because in any extension of it satisfying the equation $x_1 * (x_1 + x_2) \approx x_1$ it would happen that $a * b = a * (a + b) = a$ in it, and then this extension would not satisfy the other absorption equation, since it would happen that $a + (a * b) = a + a = b$.

In next example we shall use the following easy result.

Proposition 18. *Let Σ be a type of algebras and $\Sigma_0 \subseteq \Sigma$ a subtype of it. Let $\mathcal{K}$ be a variety of total Σ-algebras,*

$$\mathcal{E}_0 = \{\mathbf{p} \approx \mathbf{q} \in \mathrm{Eq}(\mathcal{K}) \mid \mathbf{p}, \mathbf{q}\ \Sigma_0\text{-}terms\},$$

and $\mathcal{K}_0 = \mathrm{Mod}_t(\mathcal{E}_0)$ the variety of total Σ_0-algebras defined by $\mathcal{E}_0$. If $S_w(\mathcal{K})$ is weak equational, then $S_w(\mathcal{K}_0)$ is weak equational.

Proof. Notice that, $\mathcal{E}_0$ being closed under Birkhoff's equational logic rules, $\mathcal{E}_0 = \mathrm{Eq}(\mathcal{K}_0)$. Let now $\mathbf{A}$ be a weak model of $\mathcal{E}_0$, and let $\widehat{\mathbf{A}}$ be the partial Σ-algebra whose Σ_0-reduct is $\mathbf{A}$ and where all realizations of operation symbols not belonging to Σ_0 are discrete. This Σ-algebra $\widehat{\mathbf{A}}$ is a weak model of $\mathrm{Eq}(\mathcal{K})$ and therefore, by assumption, it is a weak subalgebra of a total Σ-algebra $\mathbf{B}$ in $\mathcal{K}$. Then, $\mathbf{A}$ is a weak subalgebra of the Σ_0-reduct of $\mathbf{B}$, which belongs to $\mathcal{K}_0$. □

Example 19. Let Σ be a type of algebras with only one operation symbol $*$, which is binary, and let $\mathcal{K}$ be the class of all semigroups: i.e., the total variety of Σ-algebras defined by the equation $x_1 * (x_2 * x_3) \approx (x_1 * x_2) * x_3$. Let us see that $S_w(\mathcal{K})$ is not weak equational.

Consider the relative subalgebra $\mathbf{A}$ of the Σ-term algebra $\mathbf{T}_\Sigma(\{x_1, x_2, x_3, y_2, y_3\})$ supported on the initial segment A generated by

$$(x_1 * x_2) * x_3, (x_1 * y_2) * y_3, x_2 * x_3, y_2 * y_3,$$

and let θ be the equivalence relation on A that identifies $x_2 * x_3$ with $y_2 * y_3$. It is straightforward to prove that $\mathbf{A}$ is a weak subalgebra of the free semigroup generated by $\{x_1, x_2, x_3, y_2, y_3\}$, and that θ is a closed congruence on $\mathbf{A}$. But $\mathbf{A}/_\theta \notin S_w(\mathcal{K})$, because it does not satisfy the quasi-equation

$$\Big((x_1 * x_2) * x_3 \approx (x_1 * x_2) * x_3 \wedge (x_4 * x_5) * x_6 \approx (x_4 * x_5) * x_6 \\ \wedge x_2 * x_3 \approx x_5 * x_6\Big) \to (x_1 * x_2) * x_3 \approx (x_4 * x_5) * x_6$$

w.r.t. any valuation $v : \mathcal{X} \to A/_\theta$ defined by

$$x_1, x_4 \mapsto [x_1]_\theta,\ x_2 \mapsto [x_2]_\theta,\ x_3 \mapsto [x_3]_\theta,\ x_5 \mapsto [y_2]_\theta,\ x_6 \mapsto [y_3]_\theta.$$

Therefore, $S_w(\mathcal{K})$ is not closed under closed homomorphic images and thus it is not weak equational.

Using the last proposition, it also implies that the weak hereditary classes of the varieties of monoids and groups are not weak equational. To cover the cases of abelian groups, rings, lattices, etc., we must prove that the weak hereditary class of the variety of abelian semigroups

$$\mathcal{K}' = \mathrm{Mod}_t(x_1 * (x_2 * x_3) \approx (x_1 * x_2) * x_3, x_1 * x_2 \approx x_2 \approx x_1)$$

is not weak equational either. To do it, we can use an argument similar to the previous one.

Let $\mathcal{K}_{ab} = \mathrm{Mod}_t(x_1 * x_2 \approx x_2 * x_1)$: its weak hereditary class is clearly weak equational. Let $\mathbf{F}_{ab}$ be the $\mathcal{K}_{ab}$-free algebra generated by the set $G = \{x_1, x_2, x_3, y_2, y_3\}$, let $\mathbf{A}$ be the relative subalgebra of $\mathbf{F}_{ab}$ supported on the set A consisting of G and the elements

$$x_1 * x_2, x_1 * y_2, x_2 * x_3, y_2 * y_3, (x_1 * x_2) * x_3, (x_1 * y_2) * y_3,$$

and let θ be the equivalence relation on A that identifies $x_2 * x_3$ with $y_2 * y_3$. It is straightforward to prove that $\mathbf{A}$ is a weak subalgebra of the free abelian semigroup generated by $\{x_1, x_2, x_3, y_2, y_3\}$, and that θ is a closed congruence on $\mathbf{A}$. But $\mathbf{A}/\theta \notin S_w(\mathcal{K}')$, because it does not satisfy the quasi-equation

$$\Big((x_1 * x_2) * x_3 \approx (x_1 * x_2) * x_3 \wedge (x_4 * x_5) * x_6 \approx (x_4 * x_5) * x_6 \\ \wedge x_2 * x_3 \approx x_5 * x_6\Big) \to (x_1 * x_2) * x_3 \approx (x_4 * x_5) * x_6$$

with respect to any valuation $v : \mathcal{X} \to A/\theta$ such that

$$x_1, x_4 \mapsto [x_1]_\theta,\ x_2 \mapsto [x_2]_\theta,\ x_3 \mapsto [x_3]_\theta,\ x_5 \mapsto [y_2]_\theta,\ x_6 \mapsto [y_3]_\theta.$$

Therefore, $S_w(\mathcal{K}')$ is not closed under closed homomorphic images and hence it is not weak equational.

To close this paper, let us point out the following straightforward consequence of Corollary 11.

Corollary 20. *Let $\mathcal{K}$ be a total variety. Then, with the notations of Lemma 10, $S_w(\mathcal{K})$ is a weak equational class if and only if for every partial algebra $\mathbf{A}$, if $\varepsilon(\mathcal{K}) \cap A^2 = \Delta_A$, then $\theta(\mathcal{K}) \cap A^2 = \Delta_A$.* □

This result suggests that some sort of congruence extension property may be relevant in the characterization of those total varieties whose weak hereditary class is weak equational. The investigation of other congruence extension properties for partial algebras has found applications in other, completely different, contexts [5].

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