Midpoints as average representations of pairs of descriptions by means of fuzzy subsets

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Abstract

Let d be a metric on the set $\mathcal{FP}(X)$ of fuzzy subsets of a set X. A midpoint of two fuzzy subsets $\mu, \nu \in \mathcal{FP}(X)$ is any fuzzy subset $\xi \in \mathcal{FP}(X)$ such that $d(\xi, \mu) = d(\xi, \nu) = \frac{1}{2}d(\mu, \nu)$. These midpoints can be used to describe "middle ways" or "compromises" between two situations described by the fuzzy subsets μ and ν. In this work we explicitly compute midpoints for weighted Hamming distances and for weighted maximum distances. The former is a generalization of a previous work by Nieto and Torres (Artif. Intell. Med. 27 (2003), 81-101). We also propose a new application of midpoints in medicine, based on their use as average representations of patients of which we have available two descriptions as fuzzy subsets of a set of attribute variables.

1 Introduction

There are many situations where it is useful to discern the fuzzy subsets that can be considered as "middle ways" or "compromises" between two given fuzzy subsets of a given finite set. For instance, in order to get patterns in settings such as drug consumption among teenagers, one can consider fuzzy variables like smoking, alcohol drinking, parental abuse, etc.: each sampled individual can be represented then by means of a fuzzy subset of the universe of attribute variables under consideration, and, in this context, compromises are relevant to reach agreements, to take decisions or to compare two or more situations [6]. A patient's symptoms profile can also be described by means of such a fuzzy set, and the ensemble of middle ways between two such descriptions of a given patient provided by two independent raters can be used as a new representation of the patient; cf. Section 5. Such middle ways can also be used to capture an "average" of two uncertain descriptions of biomolecules, for instance two fuzzy genomes in the sense of Sadegh-Zadeh [7] or two profiles derived from multiple alignments; we hope to report on this kind of applications in the near future.

These middle ways between two fuzzy subsets are formalized by means of midpoints [6]. A midpoint of two fuzzy subsets μ, ν of a finite set X is any fuzzy subset ξ of X whose distance to μ and ν is exactly half the distance between these two fuzzy subsets. Of course, the actual content, and the range of application, of this definition depends on the chosen distance. For instance, if we consider the euclidean distance $d(\mu, \nu) = \sqrt{\sum_{x\in X}(\mu(x) - \nu(x))^2}$, then we know from euclidean geometry that any two fuzzy subsets μ and ν of X have one, and only one, midpoint: namely, $(\mu + \nu)/2$. For other distances, like the one introduced by Nieto et al. to compare polynucleotides [5], there exist pairs of fuzzy subsets without midpoints [8, §5]. Finally, there are distances with respect to which most pairs of fuzzy subsets have infinitely many midpoints. Two popular and meaningful metrics used in fuzzy mathematics that, in general, yield infinite sets of midpoints are the Hamming and maximum distances.

Nieto and Torres [6] recently introduced the midpoints with respect to Hamming distance and they discussed several applications in medicine. In that paper they only gave an explicit description of these midpoints for fuzzy subsets of 2-element sets, mainly because for greater sets the case-based description they derived was too complicated. In Section 3 we revisit their work, and we give a very succinct description of the midpoints of two fuzzy subsets for weighted Hamming distances on finite sets with an arbitrary number of elements in terms of the intersection of a linear variety and a rectangular parallelepiped in Kosko's hypercube $[0,1]^n$, and we establish several properties for them. Then, in Section 4 we do the same for weighted maximum distances, obtaining a similar, but simpler, representation.

Finally, in Section 5 we discuss a new application in medicine of the sets of midpoints with respect to a distance. It often happens that one has available two descriptions of a given patient as a fuzzy subset of a set of attribute variables. If it happens in a context where a certain metric is used to compare patients with one another or with some reference vector, then the set of midpoints of both descriptions of a patient with respect to that distance can be used as a new, "average" representation of the patient to be employed in comparisons. In this paper we show a specific instance of the application of midpoints in this sense; other examples will appear elsewhere [1].

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2 Preliminaries

Let us fix from now on a finite set

$$X = \{x_1, \ldots, x_n\},$$

and let $\mathcal{FP}(X)$ denote the set of its $[0,1]$-valued fuzzy subsets. To simplify the notations, given a fuzzy subset $\mu, \nu, \ldots$ of X, we shall write $\mu_i, \nu_j, \ldots$ instead of $\mu(x_i), \nu(x_j), \ldots$.

The mapping sending every $\mu \in \mathcal{FP}(X)$ to the vector $(\mu_1, \ldots, \mu_n) \in [0,1]^n$ is a bijection $\mathcal{FP}(X) \cong [0,1]^n$ that identifies, in a one-to-one way, every fuzzy subset of X with a vector of Kosko's n-dimensional hypercube. This allows to translate distances on $[0,1]^n$ into distances on $\mathcal{FP}(X)$.

Let now d be any distance on $\mathcal{FP}(X)$. Given $\mu, \nu \in \mathcal{FP}(X)$, a fuzzy subset $\xi \in \mathcal{FP}(X)$ is a *midpoint* of μ and ν with respect to d if and only if

$$d(\xi, \mu) = d(\xi, \nu) = \frac{1}{2} d(\mu, \nu).$$

Let $\mathrm{mid}_d(\mu, \nu) \subseteq \mathcal{FP}(X)$ denote the set of all midpoints of μ and ν with respect to d.

For instance, it is well known that if

$$d(\mu, \nu) = \sqrt{\sum_{i=1}^n (\mu_i - \nu_i)^2}$$

is the euclidean distance on $\mathcal{FP}(X)$ (corresponding to the euclidean distance on $[0,1]^n$), then, for every $\mu, \nu \in \mathcal{FP}(X)$,

$$\mathrm{mid}_d(\mu, \nu) = \{\frac{1}{2}(\mu + \nu)\}.$$

3 Midpoints for weighted Hamming distances

Let $\omega = (\omega_1, \ldots, \omega_n) \in (\mathbb{R}^+)^n$ be any vector of positive weights. The ω*-weighted Hamming* distance on $\mathcal{FP}(X)$ is defined, for every $\mu, \nu \in \mathcal{FP}(X)$, by

$$d_{H,\omega}(\mu, \nu) = \sum_{i=1}^n \omega_i |\mu_i - \nu_i|.$$

Let us denote by $\mathrm{mid}_{H,\omega}$ the sets of midpoints with respect to $d_{H,\omega}$. Thus, for every $\mu, \nu, \xi \in \mathcal{FP}(X)$, $\xi \in \mathrm{mid}_{H,\omega}(\mu, \nu)$ if and only if

$$\begin{aligned} \textstyle\sum_{i=1}^n \omega_i |\mu_i - \xi_i| &= \textstyle\sum_{i=1}^n \omega_i |\nu_i - \xi_i| \\ &= \textstyle\frac{1}{2} \sum_{i=1}^n \omega_i |\mu_i - \nu_i|. \end{aligned}$$

The following result provides a concise explicit description of $\mathrm{mid}_{H,\omega}(\mu, \nu)$.

Theorem 1 *For every $(\mu,\nu) \in \mathcal{FP}(X)^2$, let*

$$I^{(+)} = \{i \mid \mu_i < \nu_i\},\ I^{(-)} = \{i \mid \mu_i > \nu_i\}.$$

Then, for every $\mu,\nu,\xi \in \mathcal{FP}(X)$,

$$\xi \in \mathrm{mid}_{H,\omega}(\mu,\nu)$$

if and only if it satisfies the following two conditions:

1) $\mu_i \wedge \nu_i \leq \xi_i \leq \mu_i \vee \nu_i$ for every i.

2) $\sum_{i\in I^{(+)}} \omega_i(\xi_i - \frac{\mu_i+\nu_i}{2}) = \sum_{i\in I^{(-)}} \omega_i(\xi_i - \frac{\mu_i+\nu_i}{2})$.

Proof. If $\xi \in \mathrm{mid}_{H,\omega}(\mu,\nu)$, then

$$\sum_{i=1}^{n} \omega_i(|\mu_i - \xi_i| + |\nu_i - \xi_i| - |\mu_i - \nu_i|) = 0.$$

Since all addends

$$\omega_i(|\mu_i - \xi_i| + |\nu_i - \xi_i| - |\mu_i - \nu_i|)$$

in this sum are non-negative, it adds up 0 if and only if all these addends are 0, i.e., if and only if

$$|\mu_i - \xi_i| + |\nu_i - \xi_i| - |\mu_i - \nu_i| = 0$$

for every $i = 1,\ldots,n$, and this is equivalent to

$$\mu_i \wedge \nu_i \leq \xi_i \leq \mu_i \vee \nu_i$$

for every $i = 1,\ldots,n$. This proves that every $\xi \in \mathrm{mid}_{H,\omega}(\mu,\nu)$ satisfies condition (1) in the statement.

On the other hand, if ξ satisfies (1), then

$$\begin{aligned} d_{H,\omega}(\mu,\xi) &= \textstyle\sum_{i\in I^{(+)}} \omega_i(\xi_i - \mu_i) \\ &\quad + \textstyle\sum_{i\in I^{(-)}} \omega_i(\mu_i - \xi_i) \\ d_{H,\omega}(\nu,\xi) &= \textstyle\sum_{i\in I^{(+)}} \omega_i(\nu_i - \xi_i) \\ &\quad + \textstyle\sum_{i\in I^{(-)}} \omega_i(\xi_i - \nu_i) \end{aligned}$$

(notice that if $i \notin I^{(+)} \cup I^{(-)}$, then $\mu_i = \nu_i$ and hence, if ξ satisfies (1), $\xi_i = \mu_i = \nu_i$). Now, if $\xi \in \mathrm{mid}_{H,\omega}(\mu,\nu)$, then

$$d_{H,\omega}(\mu,\xi) = d_{H,\omega}(\nu,\xi)$$

and if ξ satisfies (1), this is equivalent to

$$\begin{aligned} &\textstyle\sum_{i\in I^{(+)}} \omega_i(\xi_i - \mu_i) + \sum_{i\in I^{(-)}} \omega_i(\mu_i - \xi_i) \\ &= \textstyle\sum_{i\in I^{(+)}} \omega_i(\nu_i - \xi_i) + \sum_{i\in I^{(-)}} \omega_i(\xi_i - \nu_i), \end{aligned}$$

and a simple computation shows this equation to be equivalent to condition (2). This proves that conditions (1) and (2) are necessary for ξ being a midpoint of μ and ν with respect to $d_{H,\omega}$.

Conversely, if $\xi \in \mathcal{FP}(X)$ satisfies conditions (1) and (2), then in the proof of the previous implication we have seen that

$$d_{H,\omega}(\mu,\xi) = d_{H,\omega}(\nu,\xi)$$

and that

$$d_{H,\omega}(\mu,\xi) + d_{H,\omega}(\nu,\xi) - d_{H,\omega}(\mu,\nu) = 0,$$

which clearly entails that $\xi \in \mathrm{mid}_{H,\omega}(\mu,\nu)$. ∎

Notice that if we interchange the roles of μ and ν in the definition of $I^{(+)}$ and $I^{(-)}$, this simply interchanges these two sets and it does not alter the equation in condition (2).

Thus, the set of those points in $[0,1]^n$ corresponding to elements of $\mathrm{mid}_{H,\omega}(\mu,\nu)$ is equal to the intersection of the linear variety defined by the equations

$$x_i = \mu_i \quad \text{for every } i \notin I^{(+)} \cup I^{(-)}$$

$$\begin{aligned} &\textstyle\sum_{i\in I^{(+)}} \omega_i x_i - \sum_{i\in I^{(-)}} \omega_i x_i \\ &\quad = \textstyle\sum_{i\in I^{(+)}} \omega_i \frac{\mu_i+\nu_i}{2} - \sum_{i\in I^{(-)}} \omega_i \frac{\mu_i+\nu_i}{2} \end{aligned}$$

with the rectangular parallelepiped

$$\prod_{i=1}^{n} [m_i, M_i],$$

where

$$m_i = \begin{cases} \mu_i & \text{if } i \in I^{(+)} \\ \nu_i & \text{if } i \in I^{(-)} \\ 0 & \text{otherwise} \end{cases}$$

$$M_i = \begin{cases} \nu_i & \text{if } i \in I^{(+)} \\ \mu_i & \text{if } i \in I^{(-)} \\ 1 & \text{otherwise} \end{cases}$$

Example 2 *Let $n = 3$, and let μ and ν be the fuzzy subsets of $X = \{x_1,x_2,x_3\}$ corresponding to the points $(0.4,0.3,0.5)$ and $(0.8,0.3,0.3)$ of $[0,1]^3$. Set, on the other hand, $\omega = (1,2,4)$.*

Given any $\xi \in \mathcal{FP}(X)$, condition (1) in Theorem 1 says in this case

$$\begin{array}{l} 0.4 \leq \xi_1 \leq 0.8 \\ \xi_2 = 0.3 \\ 0.3 \leq \xi_3 \leq 0.5 \end{array}$$

As far as condition (2), we have that $I^{(+)} = \{1\}$ and $I^{(-)} = \{3\}$ and hence it corresponds to the equation

$$1 \cdot (\xi_1 - \frac{0.4+0.8}{2}) = 4 \cdot (\xi_3 - \frac{0.5+0.3}{2}).$$

Hence, in all, $\mathrm{mid}_{H,\omega}(\mu,\nu)$ consists of the fuzzy subsets ξ of X corresponding to the points (x,y,z) in $[0,1]^3$ such that

$$\begin{array}{l} 0.4 \leq x \leq 0.8 \\ 0.3 \leq z \leq 0.5 \\ y = 0.3 \\ z = \frac{1}{4}(x+1) \end{array}$$

This is the line segment with endpoints $(0.4, 0.3, 0.35)$ and $(0.8, 0.3, 0.45)$.

Example 3 *Let 1_X denote the crisp set X and 0_X the empty set: they correspond to $(1,\dots,1)$ and $(0,\dots,0)$ in $[0,1]^n$, respectively. Let $\omega = (\omega_1,\dots,\omega_n)$ be any vector of positive weights.*

Consider the pair $(1_X, 0_X) \in \mathcal{FP}(X)$. Then, $I^{(+)} = \emptyset$ and $I^{(-)} = \{1,\dots,n\}$, and therefore $\mathrm{mid}_{H,\omega}(1_X, 0_X)$ consists of those fuzzy subsets $\xi \in \mathcal{FP}(X)$ such that

$$\sum_{i=1}^{n} \omega_i(\xi_i - \frac{1}{2}) = 0.$$

These fuzzy subsets correspond to the intersection of $[0,1]^n$ with the hyperplane

$$\sum_{i=1}^{n} \omega_i x_i = \frac{1}{2}\sum_{i=1}^{n} \omega_i$$

orthogonal to ω through $(1/2,\dots,1/2)$.

Let now $X = X_1 \sqcup X_2$ be any partition of X into two non-empty disjoint crisp subsets, and let us understand these subsets as fuzzy subsets $1_{X_1}, 1_{X_2}$ of X. If we consider the pair $(1_{X_1}, 1_{X_2}) \in \mathcal{FP}(X)$, then

$$I^{(+)} = \{i \mid x_i \in X_2\},\ I^{(-)} = \{i \mid x_i \in X_1\}$$

and thus $\mathrm{mid}_{H,\omega}(1_{X_1}, 1_{X_2})$ consists of those fuzzy subsets $\xi \in \mathcal{FP}(X)$ such that

$$\sum_{x_i \in X_1} \omega_i(\xi_i - \frac{1}{2}) - \sum_{x_i \in X_2} \omega_i(\xi_i - \frac{1}{2}) = 0.$$

In particular the sets $\mathrm{mid}_{H,\omega}(1_X, 0_X)$ and $\mathrm{mid}_{H,\omega}(1_{X_1}, 1_{X_2})$ are different.

This shows that if we interchange one or several images between two fuzzy subsets of X, the set of midpoints with respect to the weighted Hamming distance may vary. This does not happen with respect to the euclidean distance.

It is clear from conditions (1) and (2) in Theorem 1 that the midpoint $\frac{1}{2}(\mu+\nu)$ of μ and ν with respect to the euclidean distance always belongs to $\mathrm{mid}_{H,\omega}(\mu,\nu)$, and in particular that this set is non-empty. But, in general, this set of midpoints will be infinite, as the following result shows.

Corollary 4 *For every $\mu, \nu \in \mathcal{FP}(X)$,*

$$\mathrm{mid}_{H,\omega}(\mu,\nu) = \{\frac{1}{2}(\mu+\nu)\}$$

if there exists at most one $i_0 \in \{1,\dots,n\}$ such that $\mu_{i_0} \neq \nu_{i_0}$, and $\mathrm{mid}_{H,\omega}(\mu,\nu)$ is an infinite set otherwise.

Proof. Assume first that $\mu_i = \nu_i$ for every $i \neq i_0$, and let $\xi \in \mathrm{mid}_{H,\omega}(\mu,\nu)$. Then, condition (1) in Theorem 1 implies that, for every $i \neq i_0$, $\xi_i = \mu_i = (\mu_i + \nu_i)/2$. Now, if $\mu_{i_0} = \nu_{i_0}$ too, then condition (1) also implies that $\xi_{i_0} = (\mu_{i_0} + \nu_{i_0})/2$, while if $\mu_{i_0} \neq \nu_{i_0}$, then condition (2) says $\omega_{i_0}\xi_{i_0} = \omega_{i_0}(\mu_{i_0} + \nu_{i_0})/2$ and then, since $\omega_{i_0} > 0$, $\xi_{i_0} = (\mu_{i_0} + \nu_{i_0})/2$, too.

Let us assume now that μ and ν take different values on at least two elements of X. Without any loss of generality we may assume that $\mu_1 < \nu_1$ and $\mu_2 \neq \nu_2$. Then, by Theorem 1, there exists a bijection between the set of those $\xi \in \mathrm{mid}_{H,\omega}(\mu,\nu)$ such that $\xi_i = (\mu_i + \nu_i)/2$ for every $i \geq 3$ and the set of points (ξ_1, ξ_2) in the non-trivial rectangle

$$[\mu_1, \nu_1] \times [\mu_2 \wedge \nu_2, \mu_2 \vee \nu_2]$$

that lie on the line of equation

$$y - \frac{\mu_2 + \nu_2}{2} = \pm\frac{\omega_1}{\omega_2}(x - \frac{\mu_1 + \nu_1}{2});$$

the sign $+$ corresponds to the case $\mu_2 > \nu_2$, and the sign $-$ to $\mu_2 < \nu_2$. Since the center $(\frac{\mu_1+\nu_1}{2}, \frac{\mu_2+\nu_2}{2})$ of the rectangle belongs to this line, these points form a non-trivial line segment, and in particular there are infinitely many of them. ∎

4 Midpoints for weighted maximum distances

Let $\omega = (\omega_1, \ldots, \omega_n) \in (\mathbb{R}^+)^n$ be again any vector of positive weights. The *ω-weighted maximum* distance on $\mathcal{FP}(X)$ is defined, for every $\mu, \nu \in \mathcal{FP}(X)$, by

$$d_{\infty,\omega}(\mu, \nu) = \bigvee_{i=1}^{n} \omega_i |\mu_i - \nu_i|.$$

Let us denote by $\mathrm{mid}_{\infty,\omega}$ the sets of midpoints with respect to this metric. Then, for every $\mu, \nu, \xi \in FP(X)$, $\xi \in \mathrm{mid}_{\infty,\omega}(\mu, \nu)$ if and only if

$$\begin{aligned} \textstyle\bigvee_{i=1}^{n} \omega_i |\mu_i - \xi_i| &= \textstyle\bigvee_{i=1}^{n} \omega_i |\nu_i - \xi_i| \\ &= \textstyle\frac{1}{2} \bigvee_{i=1}^{n} \omega_i |\mu_i - \nu_i|. \end{aligned}$$

The following result provides a concise explicit description of $\mathrm{mid}_{\infty,\omega}(\mu, \nu)$.

Theorem 5 *For every $\omega \in (\mathbb{R}^+)^n$ and for every $\mu, \nu, \xi \in \mathcal{FP}(X)$, $\xi \in \mathrm{mid}_{\infty,\omega}(\mu, \nu)$ if and only if it satisfies the following condition: for every $i = 1, \ldots, n$,*

$$\begin{aligned} (\mu_i \vee \nu_i) - \tfrac{1}{2\omega_i} d_{\infty,\omega}(\mu, \nu) &\leq \xi_i \\ &\leq (\mu_i \vee \nu_i) + \tfrac{1}{2\omega_i} d_{\infty,\omega}(\mu, \nu). \end{aligned}$$

Proof. Let us assume that $d_{\infty,\omega}(\mu, \nu) = \omega_{i_0} |\mu_{i_0} - \nu_{i_0}|$ and that $\mu_{i_0} \leq \nu_{i_0}$, so that, actually, $d_{\infty,\omega}(\mu, \nu) = \omega_{i_0}(\nu_{i_0} - \mu_{i_0})$. In this case, if $\xi \in \mathcal{FP}(X)$ is such that

$$d_{\infty,\omega}(\xi, \mu) = d_{\infty,\omega}(\xi, \nu) = \frac{1}{2} d_{\infty,\omega}(\mu, \nu),$$

then $\xi_{i_0} = (\mu_{i_0} + \nu_{i_0})/2$. Indeed, if $\xi_{i_0} > (\mu_{i_0} + \nu_{i_0})/2$, then

$$\omega_{i_0} |\mu_{i_0} - \xi_{i_0}| > \omega_{i_0} (\nu_{i_0} - \mu_{i_0})/2,$$

and hence $d_{\infty,\omega}(\xi, \mu) > \frac{1}{2} d_{\infty,\omega}(\mu, \nu)$, while if $\xi_{i_0} < (\mu_{i_0} + \nu_{i_0})/2$, then

$$\omega_{i_0} |\nu_{i_0} - \xi_{i_0}| > \omega_{i_0} (\nu_{i_0} - \mu_{i_0})/2,$$

and hence $d_{\infty,\omega}(\xi, \nu) > \frac{1}{2} d_{\infty,\omega}(\mu, \nu)$, too.

Now notice that if $\xi_{i_0} = (\mu_{i_0} + \nu_{i_0})/2$, then

$$\begin{aligned} \omega_{i_0} |\mu_{i_0} - \xi_{i_0}| &= \omega_{i_0} |\nu_{i_0} - \xi_{i_0}| \\ &= \omega_{i_0} |\nu_{i_0} - \mu_{i_0}|/2 \\ &= d_{\infty,\omega}(\mu, \nu)/2. \end{aligned}$$

Therefore, to impose that

$$d_{\infty,\omega}(\xi, \mu) = d_{\infty,\omega}(\xi, \nu) = \frac{1}{2} d_{\infty,\omega}(\mu, \nu)$$

is equivalent to impose that

$$\omega_i |\mu_i - \xi_i|, \omega_i |\nu_i - \xi_i| \leq \frac{1}{2} \omega_{i_0} |\nu_{i_0} - \mu_{i_0}|,$$

i.e., that

$$\begin{aligned} \mu_i \vee \nu_i - \tfrac{\omega_{i_0}}{2\omega_i} |\mu_{i_0} - \nu_{i_0}| &\leq \xi_i \\ &\leq \mu_i \wedge \nu_i + \tfrac{\omega_{i_0}}{2\omega_i} |\mu_{i_0} - \nu_{i_0}| \end{aligned}$$

for every $i \neq i_0$, This, together with $\xi_{i_0} = (\mu_{i_0} + \nu_{i_0})/2$, is equivalent to the condition in the statement. ∎

Notice that, for every $i = 1, \ldots, n$ such that $\omega_i |\mu_i - \nu_i| = d_{\infty,\omega}(\mu, \nu)$, the interval

$$\Big[\mu_i \vee \nu_i - \tfrac{1}{2\omega_i} d_{\infty,\omega}(\mu, \nu), \\ \mu_i \wedge \nu_i + \tfrac{1}{2\omega_i} d_{\infty,\omega}(\mu, \nu)\Big]$$

reduces to the singleton $\{(\mu_i + \nu_i)/2\}$. On the other hand, in the general case, this interval does not change if we interchange μ_i and ν_i. This shows that, as it happens with the euclidean distance and against what happens with (weighted) Hamming distances, if we interchange one or several images between two fuzzy subsets of X, their set of midpoints with respect to a weighted maximum distance does not vary.

Given $\mu, \nu \in \mathcal{FP}(X)$, let

$$I_{\infty,\omega} = \{i \mid \omega_i |\mu_i - \nu_i| = d_{\infty,\omega}(\mu, \nu)\}.$$

Then, the set of those points in $[0, 1]^n$ corresponding to elements of $\mathrm{mid}_{\infty,\omega}(\mu, \nu)$ is given by the rectangular parallelepiped

$$\textstyle\prod_{i \notin I_{\infty,\omega}} \Big[\Big(\mu_i \vee \nu_i - \frac{1}{2\omega_i} d_{\infty,\omega}(\mu, \nu)\Big) \vee 0, \\ \Big(\mu_i \wedge \nu_i + \frac{1}{2\omega_i} d_{\infty,\omega}(\mu, \nu)\Big) \wedge 1\Big]$$

on the linear variety

$$x_i = \frac{1}{2}(\mu_i + \nu_i), \quad i \in I_{\infty,\omega}.$$

Notice moreover that $I_{\infty,\omega} \neq \emptyset$.

Example 6 *Let n, μ, ν and ω as in Example 2 above. Then $d_{\infty,\omega}(\mu,\nu) = 0.8$ and $\mathrm{mid}_{\infty,\omega}(\mu,\nu)$ consists of the fuzzy subsets ξ of X corresponding to the points (x, y, z) in $[0,1]^3$ such that*

$$\begin{aligned} &0.4 \leq x \leq 0.8 \\ &0.1 \leq y \leq 0.5 \\ &z = 0.4 \end{aligned}$$

This is the square $[0.4, 0.8] \times [0.1, 0.5]$ on the plane $z = 0.4$.

It is easy to check that $(\mu + \nu)/2$ always belongs to $\mathrm{mid}_{\infty,\omega}(\mu,\nu)$, which implies that this set is always non-empty. Indeed, the condition

$$\begin{aligned} \mu_i \vee \nu_i - \tfrac{1}{2\omega_i} d_{\infty,\omega}(\mu,\nu) \leq \tfrac{\mu_i+\nu_i}{2} \\ \leq \mu_i \wedge \nu_i + \tfrac{1}{2\omega_i} d_{\infty,\omega}(\mu,\nu) \end{aligned}$$

is easily seen to be equivalent to

$$\omega_i|\mu_i - \nu_i| \leq d_{\infty,\omega}(\mu,\nu),$$

which holds by assumption.

Corollary 7 *For every $\mu, \nu \in \mathcal{FP}(X)$,*

$$\mathrm{mid}_{H,\omega}(\mu,\nu) = \{\frac{\mu+\nu}{2}\}$$

if $\omega_i|\mu_i - \nu_i| = \omega_j|\mu_j - \nu_j|$ for every $1 \leq i, j \leq n$, and $\mathrm{mid}_{H,\omega}(\mu,\nu)$ is an infinite set otherwise.

Proof. Notice that the segment

$$\Big[\mu_i \vee \nu_i - \tfrac{1}{2\omega_i} d_{\infty,\omega}(\mu,\nu), \\ \mu_i \wedge \nu_i + \tfrac{1}{2\omega_i} d_{\infty,\omega}(\mu,\nu)\Big]$$

reduces to a singleton if and only if

$$\begin{aligned} \mu_i \vee \nu_i - \tfrac{1}{2\omega_i} d_{\infty,\omega}(\mu,\nu) \\ = \mu_i \vee \nu_i + \tfrac{1}{2\omega_i} d_{\infty,\omega}(\mu,\nu) \end{aligned}$$

which turns out to be equivalent to

$$\omega_i(\mu_i \vee \nu_i - \mu_i \wedge \nu_i) = d_{\infty,\omega}(\mu,\nu),$$

i.e., to $\omega_i|\mu_i - \nu_i| = d_{\infty,\omega}(\mu,\nu)$. ∎

Example 8 *With the notations of Example 3, both $\mathrm{mid}_{\infty,\omega}(1_X, 0_X)$ and $\mathrm{mid}_{\infty,\omega}(1_{X_1}, 1_{X_2})$ are equal to $\{\frac{1}{2}1_X\}$.*

5 An application

In medicine, it is usual to represent a patient by means of an nth dimensional vector of numerical attributes in $[0,1]$: for instance, fit values deduced from some tests, or values describing to what extent the patient shows several symptoms. These vectors are elements of Kosko's hypercube $[0,1]^n$ and can be viewed as fuzzy subsets of the finite, n-element set of attribute variables taken into account.

In certain circumstances, one may have several such descriptions for every patient. For instance, these can be descriptions provided independently by several expert raters, or descriptions obtained at different times. In these cases, it is usual to take some kind of average, of aggregation, of all available descriptions of a patient as a working description of her/him; see, for instance, [4, Chap. 5]. In the case of two descriptions of each patient in a context where some metric is used to compare such descriptions, we postulate the use of the full set of midpoints of these descriptions with respect to this metric as a new representation of patients to be employed in comparisons. To end this paper, we show an example of such an application of midpoints, in this case for the maximum distance. Other applications in the same spirit would require weighted Hamming or maximum distances [1], depending on which metric is used in its context.

In a classical example of the use of fuzzy methods in the analysis of concomitant mechanisms for stroke [2, 3] (see also [4, §4.3 and ff.]), C. Helgason and collaborators made two stroke experts to evaluate the degree of causal efficacy attributable to each one of three diagnostic categories —blood, vascular and heart tests— in several stroke patients. This assigned to each patient two vectors in $[0,1]^3$. Table 1 reproduces some of these pairs of vectors.

Assume now that, in this situation, we want to evaluate the highest degree of causal ef-

Patient	Rater 1	Rater 2
1	$(0.9, 0.2, 0.9)$	$(0.5, 0.1, 0.6)$
3	$(0.5, 0.8, 0.5)$	$(0.4, 0.5, 0.6)$
5	$(0.6, 0.3, 0.8)$	$(0.7, 0.6, 0.6)$
8	$(0.2, 0.1, 0.8)$	$(0.6, 0.3, 0.4)$
10	$(0.7, 0.8, 0.5)$	$(0.2, 0.4, 0.8)$

Table 1: Descriptions of some patients as pairs of vectors in Kosko's cube.

ficacy of some mechanism, among the three considered ones, in a patient: i.e., how far this patient is from $(0, 0, 0)$ under the maximum distance. Then, it can be useful to represent each patient as the set of midpoints of both descriptions with respect to the maximum distance and to use these sets to evaluate in some way the (maximum) distance of the patient to $(0, 0, 0)$. Table 2 displays the regions in Kosko's cube corresponding to these sets of midpoints.

Patient	Midpoints
1	$\{0.7\} \times [0, 0.3] \times [0.7, 0.8]$
3	$[0.35, 0.55] \times \{0.65\} \times [0.45, 0.65]$
5	$[0.55, 0.75] \times \{0.45\} \times [0.65, 0.75]$
8	$\{0.4\} \times [0.1, 0.3] \times \{0.6\}$
10	$\{0.45\} \times [0.55, 0.65] \times [0.55, 0.75]$

Table 2: Sets of midpoints with respect to the maximum distance.

We can now compute, for instance, the least, the greatest and the average distance of a midpoint of both descriptions of a patient to the reference vector $(0, 0, 0)$: the average distance is computed as the integral of $\max\{x, y, z\}$ on the region of Kosko's cube corresponding to the midpoints, normalized by the measure (length or area, depending on whether it is a segment or a rectangle) of this region. Table 3 displays these values. In it, and for each sampled patient, columns (1) and (2) contain, respectively, the distance from the description provided by the first and the second raters to $(0, 0, 0)$, and columns (3), (4) and (5) contain, respectively, the least, the greatest and the average distance of a midpoint of both descriptions to $(0, 0, 0)$.

This shows, for instance, that although the distances from the descriptions provided by

Patient	(1)	(2)	(3)	(4)	(5)
1	0.9	0.6	0.7	0.8	0.75
3	0.8	0.6	0.65	0.65	0.65
5	0.8	0.6	0.65	0.75	0.708
8	0.8	0.6	0.6	0.6	0.6
10	0.8	0.8	0.55	0.75	0.658

Table 3: Distances to $(0, 0, 0)$.

raters 1 and 2 of patients 3, 5 and 8 to $(0, 0, 0)$ are, in all three cases, 0.8 and 0.6, respectively, the set of midpoints of patient 8 is closer to $(0, 0, 0)$ than the set of midpoints of patient 3, which, in turn, is closer than that of patient 5. This represents the fact that some mechanisms had more incidence as a cause of stroke in patient 5 than any mechanism in patient 8.

There are other features that should be mentioned. For instance, the average distance from the set of midpoints of both descriptions of a patient to $(0, 0, 0)$ need not have anything to do with the usual average of the distances of both descriptions to it, neither with the distance of the usual average of both descriptions to it. Notice also that, in patient 10, the least, the greatest and the average distance of a midpoint to $(0, 0, 0)$ are smaller or equal than the corresponding values for patient 5, although the distances of the raters' descriptions of the latter to $(0, 0, 0)$ are greater or equal than those of the former.

We can also use the average descriptions of two patients provided by the corresponding sets of midpoints to evaluate how far they are under the maximum distance, as a measure of the greatest difference in the degree of causality of any mechanism in the patients' condition. For instance, the distance between every midpoint of both descriptions of patient 1 and every midpoint of both descriptions of patient 8 is 0.3, and hence this is the average distance between midpoints of both descriptions of both patients. On the other hand, the average distance between a midpoint of both descriptions of patient 3 and a midpoint of both descriptions of patient 8 is 0.45. This hints that patient 8 is closer to patient 1 than to patient 3. Notice that nothing similar could have been deduced simply from the distances between first descriptions and between second descriptions of these three patients, as in this case they are the same: the distance between the first description of patient 8 and both the first descriptions of patients 1 and 3 is 0.7, and the distance between the second description of patient 8 and both the second descriptions of patients 1 and 3 is 0.2.

Let us mention that, although in this last example these average distances between midpoints are equal to the distances between the usual averages of both descriptions of each patient, it is not the case in general: for instance, the average distance between a midpoint of both descriptions of patient 8 and a midpoint of both descriptions of patient 10 is 0.45, while the distance between the usual averages of both descriptions of these two patients, $(0.4, 0.2, 0.6)$ and $(0.45, 0.6, 0.65)$, respectively, is 0.4 and the average of the distance between their first descriptions and the distance between their second descriptions is 0.5.

6 Conclusion

The set of midpoints of two fuzzy subsets of a given finite set with respect to a given metric may be a useful representation of the "middle ways" of the objects or situations described by the fuzzy subsets. In this paper we have computed explicitly the sets of midpoints of two fuzzy subsets with respect to weighted Hamming distances, generalizing previous work by Nieto and Torres, and with respect to weighted maximum distances. We have also introduced a new application of midpoints in medicine. If we have two descriptions of each of a number of patients as fuzzy subsets of a set of attribute variables, and we want to use a certain metric to compare descriptions of this kind, then it may be useful to represent each patient by means of the set of midpoints of their descriptions with respect to this metric, and to compute average distances between these "average representations" as a measure of their dissimilarity under the considered metric.

We plan to continue developing this application in the near future, by studying the actual meaning of the average distance between two sets of midpoints and its correlation with other figures, by automatizing the computation of these average distances, and by applying this technique to other situations [1]. Our current research agenda also includes the generalization of midpoints ("middle ways" of two fuzzy subsets) to *barycenters* ("middle ways" of an arbitrary finite number of fuzzy subsets), which would allow us to handle more than two descriptions of each patient.

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