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SINGLE-PUSHOUT TRANSFORMATION OF TOTAL ALGEBRAS

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ABSTRACT

We characterize the pairs of partial homomorphisms of total Σ-algebras that have a pushout in the corresponding category, for an arbitrary signature Σ. This characterization provides the application condition for the single-pushout approach to the transformation of total algebras.

Keywords: Single-pushout transformation, partial homomorphism, total algebra

1. Introduction

The algebraic approach to the transformation of structures is a computation paradigm that was invented 30 years ago. Although it was first applied to graphs, it soon showed its potential to formalize the transformation of more general objects, like hypergraphs, relational systems, unary total and partial algebras etc., yielding relevant applications in practically all computer science areas. As its name hints, in this algebraic approach to transformation the objects to be rewritten are understood as algebraic structures, and its basic rewriting step is the replacement of a subobject of an object by another object, all this being done by means of some category-theoretical construction.

Several such algebraic approaches have been introduced so far, depending on how the replacement step is formalized. The first one, introduced by H. Ehrig, M. Pfender and H. J. Schneider,[1] is the so-called *double-pushout approach* (DPO).[2] Its name comes from the fact that the basic rewriting step is modelled by means of a diagram consisting of two pushout squares.

A decade ago, M. Löwe introduced[3] a new algebraic approach, called *single-pushout* (SPO) because its basic rewriting step is a pushout in a category of partial morphisms.[4] Formally, in the SPO approach to transformation in a category $\mathcal{C}$ (whose morphisms are understood as, in some sense, partial), a *production rule* is a morphism $r : L \to R$ in this category. Such a rule can be applied to an object G through an *occurrence* $m : L \to G$ —which is usually taken to be a total morphism in $\mathcal{C}$ for practical reasons[3]— when a pushout

$$(H, m' : R \to H, r' : G \to H)$$

of r and m in $\mathcal{C}$ exists. In this case the object H is said to be *derived* from G by the application of rule $r : L \to R$ through morphism m. Notice that when H exists, it is unique up to isomorphism in $\mathcal{C}$.

The SPO approach has been thoroughly developed since its inception, and the objects which it has been applied to have suffered a steady generalization, from graphs and hypergraphs to relational systems and unary partial algebras. During the same period, there has been an increasing interest in the algebraic transformation of total and partial algebras over an arbitrary signature. The motivation for such an interest is twofold.

Firstly, there is the need to model systems with dynamic behavior, which has led to the development of the concept of Dynamic Abstract Data Types (DADT), where notions of algebraic transformation and ADT specification, requiring arbitrary signatures in a natural way, interact. Roughly speaking, a DADT consists of an ADT, corresponding to the instant structures of the DADT, together with a collection of dynamic operations which define transformations between the instances of the ADT.[5] One possible way of specifying these dynamic operations is by using algebra transformation systems: the algebras model the instant structures of the DADT and the transformation of instant structures defined by the dynamic operations is specified by means of rules in an algebra transformation system.[6,7].

A second motivation is the use of attributes in all graph grammar proposals for Software Engineering, which allow to enrich graph-like structures by means of data type information. Attributed hypergraphs are structures consisting of two components, the graphical part and the data part: the former describes the structural aspects, and the latter is devoted to data type aspects and usually consists of some total algebra of an arbitrary type. Both parts are connected by labeling operators that assign data type attributes to structural items. The SPO transformation of attributed graphs was already introduced ten years ago by M. Löwe, M. Korff and A. Wagner,[8] but they imposed a strict distinction between the structural and the data parts, as the partial homomorphisms used to define the SPO production rules had to be totally defined on the algebras of attributes. This prevents, for instance, the removal of attributes.

As these authors already pointed out,[8] a possible way to overcome this distinction is by moving to algebras and treating the transformation of attributed hypergraphs as a transformation of algebras of an arbitrary type. After recalling that pushouts in categories of partial and total algebras with partial morphisms do not always exist, their paper finished with a call to the clarification of the theory behind the SPO transformation of partial and total algebras of an arbitrary type.

During the last years, we have laid out the theoretical basis for the DPO approach to the transformation of partial and total algebras of an arbitrary type by solving the corresponding application and uniqueness problems,[9,10] and for the SPO approach to the transformation of partial algebras of an arbitrary type using different types of partial homomorphisms by characterizing which pairs of partial homomorphisms of the same type have a pushout in the corresponding category.[11,12,13] In this paper we finally focus our attention on the SPO approach to the transformation of total algebras of an arbitrary type based on the partial homomorphisms introduced by M. Löwe.[3] In our main result we give a necessary and sufficient condition for two such partial homomorphisms to have a pushout in the category $\mathrm{P\text{-}TAlg}_\Sigma$ of total algebras of some type Σ with partial homomorphisms as morphisms, and we translate it into an operational application condition for the SPO approach to the transformation in the category $\mathrm{P\text{-}TAlg}_\Sigma$. This application condition turns out to be very involved, and we show that it cannot be simplified. Thus, although it strictly increases the toolkit available in the area of the formal specification of software systems with complex states or dynamic behavior, as for instance it allows the removal of attributes in attributed graphs, we must confess that we do not see any easy way to use it in practice.

Let us finally mention that the characterization of the pairs of partial homomorphisms of total algebras of an arbitrary type that have a pushout is also relevant in the more abstract line of research in universal algebra that aims at an encyclopedic study of category-theoretical notions such as limits, colimits, etc., on different categories of partial and total algebras, initiated by the three-papers-long series[14,15,16] by P. Burmeister and B. Wojdyło. This study has already found applications in, and some times it has been motivated by, the field of the algebraic transformation of (unary) partial algebras.[3,17,18] So far,[3,19,20,21,22] all results obtained in this connection have consisted of characterizations of the signatures for which a certain category-theoretical construction relative to a certain type of morphisms always exists. This paper, together with our previous papers on quomorphisms of partial algebras,[11,12,13] means a step beyond in this study, by addressing a further aspect of this kind of problems: when a construction, say a pushout, is known not to exist for all, say, pairs of morphisms of a given type, for which such pairs does it exist? The answer to questions like this will shed more light on the properties of the different types of partial homomorphisms, and the relationship between them.

2. Preliminaries

2.1. Partial algebras

We assume the reader familiar with the basic language and methods of the theory of total and partial algebras, as presented in the first chapters of standard textbooks.[23,24] However, and to ease his or her task, we recall in this subsection some basic definitions and facts and we take the opportunity to fix some notations and conventions to be used henceforth.

Throughout this paper, $\Sigma = (S, \Omega, \eta)$ denotes a *signature* (or *type of algebras*) with set of *sorts* S, set of *operation symbols* Ω and *arity function* $\eta : \Omega \to S^* \times S$ with $\eta(\varphi) = (\omega(\varphi), \sigma(\varphi))$ for every $\varphi \in \Omega$. An operation symbol $\varphi \in \Omega$ is *n-ary* when the length of $\omega(\varphi)$ is n. We shall denote by $\Omega^{(n)}$ the set of all n-ary operation symbols in Σ and by $\Omega^{(+)}$ the set $\Omega - \Omega^{(0)}$.

A *partial Σ-algebra* is a structure $\mathbf{A} = (A, (\varphi^{\mathbf{A}})_{\varphi\in\Omega})$, where:

- $A = (A_s)_{s\in S}$ is an *S-set* (i.e., an S-indexed family of sets), called the *carrier* of the algebra. For every $s \in S$, the set A_s is called the *carrier of sort s* of $\mathbf{A}$.
- For every $\varphi \in \Omega$, $\varphi^{\mathbf{A}} : A^{\omega(\varphi)} \to A_{\sigma(\varphi)}$ is a (possibly) partial mapping, called generically an *operation* in $\mathbf{A}$ (where[*] $A^{\lambda} = \{\emptyset\}$ and $A^{s_1\cdots s_p} = A_{s_1} \times \cdots \times A_{s_p}$ for every $s_1 \ldots s_p \in S^+$). We shall denote by $\operatorname{dom} \varphi^{\mathbf{A}}$ the domain of $\varphi^{\mathbf{A}}$.

An operation $\varphi^{\mathbf{A}}$ in a partial algebra $\mathbf{A}$ is *total* when it is a total mapping. A partial Σ-algebra is *total* when all operations in it are so. When a nullary operation in an algebra $\mathbf{A}$ is total, we say that it is *defined* in $\mathbf{A}$, and we identify it with its image.

Let $\Sigma^{(+)} = (S, \Omega^{(+)}, \eta|_{\Omega^{(+)}})$ denote the signature obtained from Σ by removing its nullary operation symbols. If $\mathbf{A} = (A, (\varphi^{\mathbf{A}})_{\varphi\in\Omega})$ is a partial Σ-algebra, then $\mathbf{A}^{(+)} = (A, (\varphi^{\mathbf{A}})_{\varphi\in\Omega^{(+)}})$ will always stand for its *$\Sigma^{(+)}$-reduct*: the partial $\Sigma^{(+)}$-algebra obtained by simply omitting the nullary operations in $\mathbf{A}$.

In the rest of this paper, given a partial algebra denoted by a capital letter in boldface type (**A**, **B**, etc.), we shall always denote, usually without any further notice, its carrier by the same capital letter, but in slanted type (A, B, etc.); an operation in it by superscripting the operation symbol with the algebra's name ($\varphi^{\mathbf{A}}, \psi^{\mathbf{B}}, \ldots$); and, if necessary, its carrier of a given sort by the same capital letter in slanted type, but with the sort as a subscript (A_s, B_t, etc.). However, and in order

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[*] The symbol λ stands for the empty word.

to lighten the notations, we shall often skip all subscripts corresponding to sorts in the names of the carriers of the algebras, the components of the homomorphisms or the congruences (see below) etc., provided there is no danger of confusion.

The *relative subalgebra* of a partial Σ-algebra $\mathbf{A} = (A, (\varphi^{\mathbf{A}})_{\varphi\in\Omega})$ supported on a subset B of its carrier is the partial Σ-algebra $\mathbf{B} = (B, (\varphi^{\mathbf{B}})_{\varphi\in\Omega})$ obtained by restricting the structure of $\mathbf{A}$ to B, in the sense that for every $\varphi \in \Omega$,

$$\operatorname{dom}\varphi^{\mathbf{B}} = \{\underline{b} \in B^{\omega(\varphi)} \cap \operatorname{dom}\varphi^{\mathbf{A}} \mid \varphi^{\mathbf{A}}(\underline{b}) \in B_{\sigma(\varphi)}\}$$

and if $\underline{b} \in \operatorname{dom}\varphi^{\mathbf{B}}$, then $\varphi^{\mathbf{B}}(\underline{b}) = \varphi^{\mathbf{A}}(\underline{b})$.

Such a relative subalgebra $\mathbf{B}$ is a *closed subalgebra* of $\mathbf{A}$ when B is a *closed subset* of $\mathbf{A}$: i.e., when for every $\varphi \in \Omega$, if $\underline{b} \in B^{\omega(\varphi)} \cap \operatorname{dom}\varphi^{\mathbf{A}}$, then $\varphi^{\mathbf{A}}(\underline{b}) \in B_{\sigma(\varphi)}$.

If a relative subalgebra $\mathbf{B}$ of partial Σ-algebra $\mathbf{A}$ is total, then it is a closed subalgebra of it. Conversely, if $\mathbf{A}$ is a total algebra, then every closed subalgebra of it is also total. As it is customary in universal algebra, we shall usually refer to closed subalgebras of total algebras as simply *subalgebras*.

There always exists the least closed subset of a partial Σ-algebra $\mathbf{A}$ containing a given subset X of its carrier; it is denoted by $C_{\mathbf{A}}(X)$ and called the closed subset of $\mathbf{A}$ *generated* by X.

A *homomorphism* $f : \mathbf{A} \to \mathbf{B}$ from a partial Σ-algebra $\mathbf{A} = (A, (\varphi^{\mathbf{A}})_{\varphi\in\Omega})$ to a partial Σ-algebra $\mathbf{B} = (B, (\varphi^{\mathbf{B}})_{\varphi\in\Omega})$ is a mapping of S-sets $f : A \to B$ that preserves the operations defined in $\mathbf{A}$, in the sense that, for every $\varphi \in \Omega$, if $\underline{a} \in \operatorname{dom}\varphi^{\mathbf{A}}$, then $f(\underline{a}) \in \operatorname{dom}\varphi^{\mathbf{B}}$ and $f(\varphi^{\mathbf{A}}(\underline{a})) = \varphi^{\mathbf{B}}(f(\underline{a}))$. Such a homomorphism is *closed* when, moreover, for every $\varphi \in \Omega$, if $f(\underline{a}) \in \operatorname{dom}\varphi^{\mathbf{B}}$, then $\underline{a} \in \operatorname{dom}\varphi^{\mathbf{A}}$.

We shall use often that if $\mathbf{A}$ is a total algebra, then any homomorphism $f : \mathbf{A} \to \mathbf{B}$ is closed, and that if $\mathbf{B}$ is a total algebra and $f : \mathbf{A} \to \mathbf{B}$ is a closed homomorphism, then $\mathbf{A}$ is also total.

Taking as objects all partial Σ-algebras (resp., all total Σ-algebras) and their homomorphisms as morphisms, we obtain a category Alg_{Σ} (resp., TAlg_{Σ}).

An equivalence relation $\theta = (\theta_s)_{s\in S}$ on the carrier of a partial Σ-algebra $\mathbf{A} = (A, (\varphi^{\mathbf{A}})_{\varphi\in\Omega})$ is a *congruence* on $\mathbf{A}$ when it is compatible with the operations in $\mathbf{A}$, in the sense that for each $\varphi \in \Omega^{(+)}$, say with $\eta(\varphi) = (s_1 \ldots s_n, s)$, if $\underline{a} = (a_1, \ldots, a_n), \underline{b} = (b_1, \ldots, b_n) \in \operatorname{dom}\varphi^{\mathbf{A}}$ are such that $(a_i, b_i) \in \theta_{s_i}$ for every $i = 1, \ldots, n$, then $(\varphi^{\mathbf{A}}(\underline{a}), \varphi^{\mathbf{A}}(\underline{b})) \in \theta_s$.

Given a relation X on the carrier of a partial Σ-algebra $\mathbf{A}$, there always exists the least congruence on $\mathbf{A}$ containing X; we shall call it the congruence on $\mathbf{A}$ *generated* by X.

Given a congruence θ on a partial Σ-algebra $\mathbf{A} = (A, (\varphi^{\mathbf{A}})_{\varphi\in\Omega})$, the *quotient* algebra

$$\mathbf{A}/\theta = (A/\theta, (\varphi^{\mathbf{A}/\theta})_{\varphi\in\Omega})$$

(where $(A/\theta)_s = A_s/\theta_s$ for every $s \in S$) is defined in the following way: for every $\varphi \in \Omega$,

- if $\varphi \in \Omega^{(0)}$, then $\varphi^{\mathbf{A}/\theta}$ is defined if and only if $\varphi^{\mathbf{A}}$ is defined, and when they are both defined, $\varphi^{\mathbf{A}/\theta} = [\varphi^{\mathbf{A}}]_{\theta_{\sigma(\varphi)}}$ (the equivalence class of $\varphi^{\mathbf{A}}$ modulo $\theta_{\sigma(\varphi)}$);

- if $\omega(\varphi) = s_1 \ldots s_p \in S^+$ and $[\underline{a}] = ([a_1]_{\theta_{s_1}}, \ldots, [a_p]_{\theta_{s_p}}) \in (A/\theta)^{\omega(\varphi)}$, then $[\underline{a}] \in \operatorname{dom}\varphi^{\mathbf{A}/\theta}$ if and only if there exists $\underline{a'} = (a'_1, \ldots, a'_p) \in \operatorname{dom}\varphi^{\mathbf{A}}$ such that $(a_i, a'_i) \in \theta_{s_i}$ for every $i = 1, \ldots, p$; and if this is the case, then $\varphi^{\mathbf{A}/\theta}([\underline{a}]) = [\varphi^{\mathbf{A}}(\underline{a'})]_{\theta_{\sigma(\varphi)}}$.

We shall denote then by $nat_\theta : \mathbf{A} \to \mathbf{A}/\theta$ the *quotient homomorphism*, given by the quotient mapping

$$\begin{array}{rcll} nat_{\theta_s} : \; A_s & \to & A_s/\theta_s, & \qquad s \in S. \\ a & \mapsto & [a]_{\theta_s} & \end{array}$$

A *free completion* of a partial Σ-algebra $\mathbf{A}$ is a total Σ-algebra $\overline{\mathbf{A}}$ together with a homomorphism $\ell_{\mathbf{A}} : \mathbf{A} \to \overline{\mathbf{A}}$ such that, for every homomorphism $f : \mathbf{A} \to \mathbf{B}$ with $\mathbf{B}$ a total Σ-algebra, there exists one, and only one, homomorphism $\bar{f} : \overline{\mathbf{A}} \to \mathbf{B}$ such that $\bar{f} \circ \ell_{\mathbf{A}} = f$.

Every partial Σ-algebra $\mathbf{A}$ has a free completion $\overline{\mathbf{A}}$ such that $\ell_{\mathbf{A}} : \mathbf{A} \to \overline{\mathbf{A}}$ is the embedding of $\mathbf{A}$ as a relative subalgebra of $\overline{\mathbf{A}}$: a construction can be found in[24] §5.3. On the other hand, the free completion of a partial Σ-algebra $\mathbf{A}$ is unique up to isomorphism, in the sense that if $\ell_{\mathbf{A}} : \mathbf{A} \to \overline{\mathbf{A}}$ and $\widetilde{\ell_{\mathbf{A}}} : \mathbf{A} \to \widetilde{\mathbf{A}}$ are two free completions of $\mathbf{A}$, then there exists an isomorphism $\bar{\ell} : \overline{\mathbf{A}} \to \widetilde{\mathbf{A}}$ such that $\widetilde{\ell_{\mathbf{A}}} = \bar{\ell} \circ \ell_{\mathbf{A}}$. Therefore, whenever we consider a free completion of a partial Σ-algebra $\mathbf{A}$, we shall assume that $\mathbf{A}$ is relative subalgebra of it.

2.2. Pushouts in Alg_Σ *and* TAlg_Σ

It is well known that the categories Alg_Σ and TAlg_Σ have all pushouts, for every signature Σ. In this subsection we recall an explicit construction of them and we take again the opportunity to fix some notations that will be used in the rest of this paper, usually without any further notice.

Let $f : \mathbf{K} \to \mathbf{A}$ and $g : \mathbf{K} \to \mathbf{B}$ be two homomorphisms of partial Σ-algebras, for some signature Σ, and let $\mathbf{A}^{(+)}$ and $\mathbf{B}^{(+)}$ be the $\Sigma^{(+)}$-reducts of $\mathbf{A}$ and $\mathbf{B}$, respectively.

The *disjoint sum* of $\mathbf{A}^{(+)}$ and $\mathbf{B}^{(+)}$ is the partial $\Sigma^{(+)}$-algebra

$$\mathbf{A}^{(+)} \oplus \mathbf{B}^{(+)} = (A \sqcup B, (\varphi^{\mathbf{A}^{(+)} \oplus \mathbf{B}^{(+)}})_{\varphi \in \Omega^{(+)}})$$

with carrier the disjoint union[†] $A \sqcup B$ of the carriers of $\mathbf{A}$ and $\mathbf{B}$, and operations defined in the following way: for every $\varphi \in \Omega^{(+)}$,

$$\operatorname{dom} \varphi^{\mathbf{A}^{(+)} \oplus \mathbf{B}^{(+)}} = \operatorname{dom} \varphi^{\mathbf{A}} \sqcup \operatorname{dom} \varphi^{\mathbf{B}}$$

and if $\underline{a} \in \operatorname{dom} \varphi^{\mathbf{A}}$ (resp., $\underline{b} \in \operatorname{dom} \varphi^{\mathbf{B}}$), then $\varphi^{\mathbf{A}^{(+)} \oplus \mathbf{B}^{(+)}}(\underline{a}) = \varphi^{\mathbf{A}}(\underline{a})$ (resp., $\varphi^{\mathbf{A}^{(+)} \oplus \mathbf{B}^{(+)}}(\underline{b}) = \varphi^{\mathbf{B}}(\underline{b})$).

Let now $\theta(f,g)$ be the congruence on $\mathbf{A}^{(+)} \oplus \mathbf{B}^{(+)}$ generated by the relation $E(f,g) \cup X_0$, where, for every $s \in S$,

$$\begin{aligned} E(f,g)_s &= \{(f_s(x), g_s(x)) \mid x \in K_s\}, \\ X_{0s} &= \{(\varphi_0^{\mathbf{A}}, \varphi_0^{\mathbf{B}}) \mid \varphi_0 \in \Omega^{(0)},\ \sigma(\varphi_0) = s,\ \varphi_0^{\mathbf{A}}, \varphi_0^{\mathbf{B}} \text{ defined}\}. \end{aligned}$$

Let $\mathbf{P} = (P, (\varphi^{\mathbf{P}})_{\varphi \in \Omega})$ be the partial Σ-algebra whose $\Sigma^{(+)}$-reduct is the quotient

$$(\mathbf{A}^{(+)} \oplus \mathbf{B}^{(+)})/\theta(f,g)$$

[†] Formally, we define $A \sqcup B$ as $A \times \{1\} \cup B \times \{2\}$, but in order to simplify the notations we shall identify A and B with their images in $A \sqcup B$.

and whose nullary operations are defined as follows: for every $\varphi_0 \in \Omega^{(0)}$, $\varphi_0^{\mathbf{P}}$ is defined if and only if $\varphi_0^{\mathbf{A}}$ or $\varphi_0^{\mathbf{B}}$ are defined, in which case $\varphi_0^{\mathbf{P}}$ is the corresponding equivalence class modulo $\theta(f,g)$. In particular, $P = (A \sqcup B)/\theta(f,g)$.

Let finally $\tilde{g} : \mathbf{A} \to \mathbf{P}$ and $\tilde{f} : \mathbf{B} \to \mathbf{P}$ be the homomorphisms given by the restrictions to $\mathbf{A}$ and $\mathbf{B}$ of the quotient mapping $A \sqcup B \to (A \sqcup B)/\theta(f,g)$.

Theorem 1 *With the previous notations, the cocone*

$$(\mathbf{P}, \tilde{g} : \mathbf{A} \to \mathbf{P}, \tilde{f} : \mathbf{B} \to \mathbf{P})$$

is a pushout of f *and* g *in* Alg_Σ. ■

If f and g are closed homomorphisms, then $X_0 \subseteq E(f,g)$, because if $\varphi_0 \in \Omega^{(0)}$ and, say, $\varphi_0^{\mathbf{A}}$ is defined, then $\varphi_0^{\mathbf{K}}$ is also defined, and then on the one hand $\varphi_0^{\mathbf{A}} = f(\varphi_0^{\mathbf{K}})$ and on the other hand $\varphi_0^{\mathbf{B}}$ is also defined and equal to $g(\varphi_0^{\mathbf{K}})$.

As far as pushouts in TAlg_Σ goes, we have the following result, which is a simple consequence of the fact that free completions define an epireflection along the inclusion functor $\mathrm{TAlg}_\Sigma \hookrightarrow \mathrm{Alg}_\Sigma$; see Theorem 20 in[10] for an elementary proof of a more general result.

Theorem 2 *Let* $f : \mathbf{K} \to \mathbf{A}$ *and* $g : \mathbf{K} \to \mathbf{B}$ *be two homomorphisms of total* Σ*-algebras, let* $(\mathbf{P}, \tilde{g} : \mathbf{A} \to \mathbf{P}, \tilde{f} : \mathbf{B} \to \mathbf{P})$ *be a pushout of* f *and* g *in* Alg_Σ*, and let* $\ell_{\mathbf{P}} : \mathbf{P} \to \overline{\mathbf{P}}$ *be a free completion of* $\mathbf{P}$*. Then, the cocone*

$$(\overline{\mathbf{P}}, \ell_{\mathbf{P}} \circ \tilde{g} : \mathbf{A} \to \overline{\mathbf{P}}, \ell_{\mathbf{P}} \circ \tilde{f} : \mathbf{B} \to \overline{\mathbf{P}})$$

is a pushout of f *and* g *in* TAlg_Σ. ■

2.3. Quomorphisms

Let $\mathbf{A}$ and $\mathbf{B}$ be two partial Σ-algebras, for some signature Σ. Given a partial mapping of S-sets $f : A \to B$, let $\mathrm{Dom}\, f$ and $\mathbf{Dom}\, f$ denote, respectively, its domain and the relative subalgebra of $\mathbf{A}$ supported on it. Then, the partial mapping $f : A \to B$ is:

- a *quomorphism* from $\mathbf{A}$ to $\mathbf{B}$ when it is a homomorphism from $\mathbf{Dom}\, f$ to $\mathbf{B}$;
- a *closed quomorphism* from $\mathbf{A}$ to $\mathbf{B}$, a *c-quomorphism* for short, when it is a closed homomorphism from $\mathbf{Dom}\, f$ to $\mathbf{B}$;
- a *closed-domain quomorphism* from $\mathbf{A}$ to $\mathbf{B}$, a *cd-quomorphism* for short, when it is a quomorphism and $\mathrm{Dom}\, f$ is a closed subset of $\mathbf{A}$;
- a *closed-domain closed quomorphism* from $\mathbf{A}$ to $\mathbf{B}$, a *cdc-quomorphism* for short, when it is simultaneously a c-quomorphism and a cd-quomorphism.

The categories of all partial Σ-algebras with quomorphisms, c-quomorphisms, cd-quomorphisms, and cdc-quomorphisms as morphisms will be denoted, respectively, by $\mathrm{Q\text{-}Alg}_\Sigma$, $\mathrm{CQ\text{-}Alg}_\Sigma$, $\mathrm{CDQ\text{-}Alg}_\Sigma$, and $\mathrm{CDCQ\text{-}Alg}_\Sigma$. The first two categories have been thoroughly studied by P. Burmeister and B. Wojdyło[14,15,16] and the last two by R. Alberich *et al.*[19] Furthermore, R. Alberich and F. Rosselló[11,12,13] have characterized the pairs of quomorphisms, c-quomorphisms, cd-quomorphisms and cdc-quomorphisms that have a pushout in $\mathrm{Q\text{-}Alg}_\Sigma$, $\mathrm{CQ\text{-}Alg}_\Sigma$, $\mathrm{CDQ\text{-}Alg}_\Sigma$ and $\mathrm{CDCQ\text{-}Alg}_\Sigma$, respectively. In this paper we shall use only the characterization for cdc-quomorphisms, which we recall below; for a proof, see Theorem 5 and Corollary 1 in the last aforementioned paper.

Theorem 3 *Two cdc-quomorphisms of partial Σ-algebras $f : \mathbf{K} \to \mathbf{A}$ and $g : \mathbf{K} \to \mathbf{B}$ have a pushout in* CDCQ-Alg$_\Sigma$ *if and only if they satisfy the following three conditions:*

CDCPO1) *There exist the greatest closed subsets A' and B' of $\mathbf{A}$ and $\mathbf{B}$, respectively, such that:*

- *$f^{-1}(A') = g^{-1}(B')$; let us call K' this subset of $\mathbf{K}$;*
- *if $\mathbf{K}'$, $\mathbf{A}'$, and $\mathbf{B}'$ are the closed subalgebras of $\mathbf{K}$, $\mathbf{A}$ and $\mathbf{B}$ supported on K', A' and B', respectively, and if $f' : \mathbf{K}' \to \mathbf{A}'$ and $g' : \mathbf{K}' \to \mathbf{B}'$ are the closed homomorphisms between these algebras induced by f and g, then there exist two closed homomorphisms $p : \mathbf{A}' \to \mathbf{C}$ and $q : \mathbf{B}' \to \mathbf{C}$ to some partial Σ-algebra $\mathbf{C}$ such that $p \circ f' = q \circ g'$.*

Let $\theta(f', g')$ be the congruence on $\mathbf{A}'^{(+)} \oplus \mathbf{B}'^{(+)}$ generated by the relation

$$E(f', g') = (\{(f_s(x), g_s(x)) \mid x \in K'_s\})_{s \in S}\,.$$

CDCPO2) *For every $\varphi \in \Omega^{(+)}$,*

$$((A' \sqcup B')/\theta(f', g'))^{\omega(\varphi)} = nat_{\theta(f',g')}(A')^{\omega(\varphi)} \cup nat_{\theta(f',g')}(B')^{\omega(\varphi)}.$$

CDCPO3) *For every closed subsets A_0 and B_0 of $\mathbf{A}'$ and $\mathbf{B}'$, respectively, such that $f^{-1}(A_0) = g^{-1}(B_0)$, if θ_0 is the congruence on $\mathbf{A}_0^{(+)} \oplus \mathbf{B}_0^{(+)}$ generated by $E(f', g') \cap (A_0 \sqcup B_0)^2$, then*

$$\theta(f', g') \cap ((A_0 \sqcup B_0) \times (A' \sqcup B')) \subseteq \theta_0.$$

And if f and g satisfy conditions (CDCPO1) to (CDCPO3) and

$$(\mathbf{P}, \tilde{g}' : \mathbf{A}' \to \mathbf{P}, \tilde{f}' : \mathbf{B}' \to \mathbf{P})$$

is a pushout of $f' : \mathbf{K}' \to \mathbf{A}'$ and $g' : \mathbf{K}' \to \mathbf{B}'$ in Alg$_\Sigma$*, then the quomorphisms $\tilde{g} : \mathbf{A} \to \mathbf{P}$ and $\tilde{f} : \mathbf{B} \to \mathbf{P}$ defined by the homomorphisms $\tilde{g}'$ and $\tilde{f}'$, respectively, are cdc-quomorphisms and*

$$(\mathbf{P}, \tilde{g} : \mathbf{A} \to \mathbf{P}, \tilde{f} : \mathbf{B} \to \mathbf{P})$$

is a pushout of $f : \mathbf{K} \to \mathbf{A}$ and $g : \mathbf{K} \to \mathbf{B}$ in CDCQ-Alg$_\Sigma$. ■

3. Pushouts of partial homomorphisms of total algebras

Let $\mathbf{A} = (A, (\varphi^{\mathbf{A}})_{\varphi \in \Omega})$ and $\mathbf{B} = (B, (\varphi^{\mathbf{B}})_{\varphi \in \Omega})$ be two total Σ-algebras, for some signature Σ. A partial mapping $f : A \to B$ is a *partial homomorphism* (of total Σ-algebras) from $\mathbf{A}$ to $\mathbf{B}$ when it is a cd-quomorphism of partial Σ-algebras, i.e., when its domain $\mathrm{Dom}\, f$ is a closed subset of $\mathbf{A}$ and $f : \mathbf{Dom}\, f \to \mathbf{B}$ is a homomorphism of Σ-algebras. We shall denote by P-TAlg$_\Sigma$ the category of total Σ-algebras with partial homomorphisms as morphisms.

If $f : \mathbf{A} \to \mathbf{B}$ is a partial homomorphism, then, since $\mathbf{A}$ is total, $\mathbf{Dom}\, f$ is also total and therefore the homomorphism $f : \mathbf{Dom}\, f \to \mathbf{B}$ is closed. This shows that every partial homomorphism of total algebras is a cdc-quomorphism and *a fortiori* a c-quomorphism. On the other hand, if $f : \mathbf{A} \to \mathbf{B}$ is a c-quomorphism with $\mathbf{A}$ and $\mathbf{B}$ total Σ-algebras, then, since $f : \mathbf{Dom}\, f \to \mathbf{B}$ is a closed homomorphism and $\mathbf{B}$ is total, $\mathbf{Dom}\, f$ is also total and in particular $\mathrm{Dom}\, f$ is a closed subset of $\mathbf{A}$. This proves the following result.

Proposition 1 P-TAlg$_\Sigma$ *is a full subcategory of* CQ-Alg$_\Sigma$, CDQ-Alg$_\Sigma$, *and* CDCQ-Alg$_\Sigma$. ■

Notice on the other hand that if $\Omega \neq \emptyset$, then P-TAlg$_\Sigma$ is not a full subcategory of Q-Alg$_\Sigma$. Indeed, if $\Omega \neq \emptyset$, there always exists a total Σ-algebra $\mathbf{A}$ with a non-closed subset B, and then the quomorphism $\mathbf{A} \to \mathbf{A}$ defined by the set-theoretical identity on B is not a partial homomorphism of total algebras.

M. Löwe[3] established the following characterization of those signatures Σ such that P-TAlg$_\Sigma$ has all pushouts; for a proof, see Proposition 2.3 and Corollary 2.7 in his cited paper.

Theorem 4 P-TAlg$_\Sigma$ *has all pushouts if and only if all operation symbols in* Σ *are unary.* ■

The main goal of this section is to characterize the pairs of partial homomorphisms of total Σ-algebras that have a pushout in P-TAlg$_\Sigma$, for an arbitrary signature Σ. To begin with, we establish a lemma about pushouts of partial homomorphisms of total algebras in CDCQ-Alg$_\Sigma$.

Lemma 1 *Let* $f : \mathbf{K} \to \mathbf{A}$ *and* $g : \mathbf{K} \to \mathbf{B}$ *be two partial homomorphisms of total* Σ*-algebras that have a pushout*

$$(\mathbf{P}, \tilde{f} : \mathbf{A} \to \mathbf{P}, \tilde{g} : \mathbf{B} \to \mathbf{P})$$

in CDCQ-Alg$_\Sigma$. *Then,* $\mathbf{P}$ *is total.*

Proof. By Theorem 3 and the explicit description of pushouts in Alg$_\Sigma$ given in §2.2, for every $\varphi \in \Omega^{(+)}$

$$P^{\omega(\varphi)} = \tilde{g}(\mathrm{Dom}\,\tilde{g})^{\omega(\varphi)} \cup \tilde{f}(\mathrm{Dom}\,\tilde{f})^{\omega(\varphi)}.$$

Since $\mathbf{Dom}\,\tilde{g}$ and $\mathbf{Dom}\,\tilde{f}$ are total algebras, we have that

$$\mathrm{dom}\,\varphi^{\mathbf{Dom}\,\tilde{g}} = (\mathrm{Dom}\,\tilde{g})^{\omega(\varphi)} \text{ and } \mathrm{dom}\,\varphi^{\mathbf{Dom}\,\tilde{f}} = (\mathrm{Dom}\,\tilde{f})^{\omega(\varphi)},$$

and then

$$P^{\omega(\varphi)} = \tilde{g}(\mathrm{dom}\,\varphi^{\mathbf{Dom}\,\tilde{g}}) \cup \tilde{f}(\mathrm{dom}\,\varphi^{\mathbf{Dom}\,\tilde{f}}) \subseteq \mathrm{dom}\,\varphi^{\mathbf{P}},$$

which implies that $\varphi^{\mathbf{P}}$ is total for every $\varphi \in \Omega^{(+)}$.

On the other hand, all nullary operations are defined in $\mathbf{Dom}\,\tilde{g}$ and $\mathbf{Dom}\,\tilde{f}$, and then by construction they are also defined in $\mathbf{P}$. This finally implies that $\mathbf{P}$ is total. ■

We can establish now the main result in this paper.

Theorem 5 *Let* $f : \mathbf{K} \to \mathbf{A}$ *and* $g : \mathbf{K} \to \mathbf{B}$ *be two partial homomorphisms of total* Σ*-algebras. The following conditions are equivalent:*

i) f *and* g *have a pushout in* CDQ-Alg$_\Sigma$ *and it is given by a total* Σ*-algebra.*

ii) f *and* g *have a pushout in* CDCQ-Alg$_\Sigma$.

iii) f *and* g *have a pushout in* P-TAlg$_\Sigma$.

And when these equivalent conditions are satisfied, the pushouts of f *and* g *in* P-TAlg$_\Sigma$, *in* CDQ-Alg$_\Sigma$ *and in* CDCQ-Alg$_\Sigma$ *are equal.*

Proof. *(i)⟹(ii)* Assume that f and g have a pushout

$$(\mathbf{Q}, \hat{g} : \mathbf{A} \to \mathbf{Q}, \hat{f} : \mathbf{B} \to \mathbf{Q})$$

in CDQ-Alg$_\Sigma$ and that $\mathbf{Q}$ is a total Σ-algebra. Since $\mathbf{A}$ and $\mathbf{B}$ are total, $\hat{g}$ and $\hat{f}$ are cdc-quomorphisms. Let now $p : \mathbf{A} \to \mathbf{C}$ and $q : \mathbf{B} \to \mathbf{C}$ be two cdc-quomorphisms of partial Σ-algebras such that $p \circ f = q \circ g$. Then, by the universal property of pushouts in CDQ-Alg$_\Sigma$, there exists one, and only one, cd-quomorphism $h : \mathbf{Q} \to \mathbf{C}$ such that $h \circ \hat{g} = p$ and $h \circ \hat{f} = q$. Since $\mathbf{Q}$ is total, h is a cdc-quomorphism, and its uniqueness in CDQ-Alg$_\Sigma$ implies its uniqueness in CDCQ-Alg$_\Sigma$. This shows that $(\mathbf{Q}, \hat{g} : \mathbf{A} \to \mathbf{P}, \hat{f} : \mathbf{B} \to \mathbf{P})$ is also a pushout of f and g in CDCQ-Alg$_\Sigma$.

(ii)$\Longrightarrow$(iii) Let

$$(\mathbf{Q}, \hat{g} : \mathbf{A} \to \mathbf{Q}, \hat{f} : \mathbf{B} \to \mathbf{Q})$$

be a pushout of f and g in CDCQ-Alg$_\Sigma$: by Lemma 1, $\mathbf{Q}$ is total and hence, by definition, $\hat{g}$ and $\hat{f}$ are partial homomorphisms of total algebras. Let now $p : \mathbf{A} \to \mathbf{C}$ and $q : \mathbf{B} \to \mathbf{C}$ be two partial homomorphisms of total Σ-algebras such that $p \circ f = q \circ g$. In particular, they are cdc-quomorphisms and therefore, by the universal property of pushouts in CDCQ-Alg$_\Sigma$, there exists one, and only one, cdc-quomorphism, i.e., one, and only one partial homomorphism of total Σ-algebras $h : \mathbf{Q} \to \mathbf{C}$ such that $h \circ \hat{g} = p$ and $h \circ \hat{f} = q$. This shows that the pushout of f and g in CDCQ-Alg$_\Sigma$ is also their pushout in P-TAlg$_\Sigma$.

(iii)$\Longrightarrow$(i) Let

$$(\mathbf{Q}, \hat{g} : \mathbf{A} \to \mathbf{Q}, \hat{f} : \mathbf{B} \to \mathbf{Q})$$

be a pushout of f and g in P-TAlg$_\Sigma$. Set $A' = \operatorname{Dom} \hat{g}$, $B' = \operatorname{Dom} \hat{f}$ and $K' = f^{-1}(A') = g^{-1}(B')$, let $\mathbf{K}'$, $\mathbf{A}'$ and $\mathbf{B}'$ be the subalgebras of $\mathbf{K}$, $\mathbf{A}$ and $\mathbf{B}$ supported on their closed subsets K', A' and B', respectively, let $f' : \mathbf{K}' \to \mathbf{A}'$ and $g' : \mathbf{K}' \to \mathbf{B}'$ be the homomorphisms between these total algebras induced by f and g, and let $\hat{g}' : \mathbf{A}' \to \mathbf{Q}$ and $\hat{f}' : \mathbf{B}' \to \mathbf{Q}$ denote the homomorphisms that define the partial homomorphisms $\hat{g} : \mathbf{A} \to \mathbf{Q}$ and $\hat{f} : \mathbf{B} \to \mathbf{Q}$.

Let now

$$(\mathbf{P}_0, \tilde{g} : \mathbf{A}' \to \mathbf{P}_0, \tilde{f} : \mathbf{B}' \to \mathbf{P}_0)$$

be a pushout of $f' : \mathbf{K}' \to \mathbf{A}'$ and $g' : \mathbf{K}' \to \mathbf{B}'$ in Alg$_\Sigma$ and let $\ell_{\mathbf{P}_0} : \mathbf{P}_0 \hookrightarrow \overline{\mathbf{P}}_0$ be a free completion of $\mathbf{P}_0$ given by an embedding of $\mathbf{P}_0$ into $\overline{\mathbf{P}}_0$ as a relative subalgebra. By Theorem 2,

$$(\mathbf{P}_0, \ell_{\mathbf{P}_0} \circ \tilde{g} : \mathbf{A}' \to \overline{\mathbf{P}}_0, \ell_{\mathbf{P}_0} \circ \tilde{f} : \mathbf{B}' \to \overline{\mathbf{P}}_0)$$

is a pushout in TAlg$_\Sigma$ of f' and g'.

The homomorphisms $\ell_{\mathbf{P}_0} \circ \tilde{g} : \mathbf{A}' \to \overline{\mathbf{P}}_0$ and $\ell_{\mathbf{P}_0} \circ \tilde{f} : \mathbf{B}' \to \overline{\mathbf{P}}_0$ define two partial homomorphisms of total Σ-algebras

$$\tilde{g}_0 : \mathbf{A} \to \overline{\mathbf{P}}_0, \quad \tilde{f}_0 : \mathbf{B} \to \overline{\mathbf{P}}_0.$$

Now, the equalities $\ell_{\mathbf{P}_0} \circ \tilde{g} \circ f' = \ell_{\mathbf{P}_0} \circ \tilde{f} \circ g'$ and $f^{-1}(\operatorname{Dom} \tilde{g}_0) = K' = g^{-1}(\operatorname{Dom} \tilde{f}_0)$ imply that $\tilde{g}_0 \circ f = \tilde{f}_0 \circ g$ as partial homomorphisms from $\mathbf{K}$ to $\overline{\mathbf{P}}_0$. Then, by the universal property of pushouts in P-TAlg$_\Sigma$, there exists one, and only one, partial homomorphism $h : \mathbf{Q} \to \overline{\mathbf{P}}_0$ such that $\tilde{g}_0 = h \circ \hat{g}$ and $\tilde{f}_0 = h \circ \hat{f}$. If we restrict these equalities to A' and B', we obtain the equalities of homomorphisms $\ell_{\mathbf{P}_0} \circ \tilde{g} = h \circ \hat{g}'$ and $\ell_{\mathbf{P}_0} \circ \tilde{f} = h \circ \hat{f}'$.

On the other hand, since $\hat{g}' : \mathbf{A}' \to \mathbf{Q}$ and $\hat{f}' : \mathbf{B}' \to \mathbf{Q}$ are two homomorphisms of total Σ-algebras such that $\hat{g}' \circ f' = \hat{f}' \circ g'$, the universal property of pushouts in TAlg$_\Sigma$ implies the existence of one, and only one, homomorphism $\bar{h} : \overline{\mathbf{P}}_0 \to \mathbf{Q}$ such that $\bar{h} \circ (\ell_{\mathbf{P}_0} \circ \tilde{g}) = \hat{g}'$ and $\bar{h} \circ (\ell_{\mathbf{P}_0} \circ \tilde{f}) = \hat{f}'$. Since $\operatorname{Dom} \tilde{g}_0 = A' = \operatorname{Dom} \hat{g}$ and $\operatorname{Dom} \tilde{f}_0 = B' = \operatorname{Dom} \hat{f}$, these equalities yield the equalities of partial homomorphisms of total algebras $\bar{h} \circ \tilde{g}_0 = \hat{g}$ and $\bar{h} \circ \tilde{f}_0 = \hat{f}$ in P-TAlg$_\Sigma$.

Summarizing, we have the following two commutative diagrams:

$$\begin{tikzcd}
\mathbf{K} \arrow[r, "f"] \arrow[d, "g"'] & \mathbf{A} \arrow[d, "\hat{g}"] \arrow[ddr, bend left, "\tilde{g}_0"] & \\
\mathbf{B} \arrow[r, "\hat{f}"'] \arrow[drr, bend right, "\tilde{f}_0"'] & \mathbf{Q} \arrow[dr, bend left, "h"] & \\
 & & \overline{\mathbf{P}}_0 \arrow[ul, bend left, "\bar{h}"]
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbf{K}' \arrow[r, "f'"] \arrow[d, "g'"'] & \mathbf{A}' \arrow[d, "\hat{g}'"] \arrow[ddr, bend left, "\ell_{\mathbf{P}_0}\circ\tilde{g}"] & \\
\mathbf{B}' \arrow[r, "\hat{f}'"'] \arrow[drr, bend right, "\ell_{\mathbf{P}_0}\circ\tilde{f}"'] & \mathbf{Q} \arrow[dr, bend left, "h"] & \\
 & & \overline{\mathbf{P}}_0 \arrow[ul, bend left, "\bar{h}"]
\end{tikzcd}$$

Working in the left-hand side diagram we obtain the equalities in P-TAlg$_\Sigma$

$$\bar{h}\circ h\circ\hat{g} = \bar{h}\circ\tilde{g}_0 = \hat{g}, \quad \bar{h}\circ h\circ\hat{f} = \bar{h}\circ\tilde{f}_0 = \hat{f}$$

which imply, by the universal property of pushouts in P-TAlg$_\Sigma$, that $\bar{h}\circ h = \mathrm{Id}_{\mathbf{Q}}$. In particular, h is a homomorphism. Then, working in the right-hand side diagram given above, we obtain the equalities in TAlg$_\Sigma$

$$h\circ\bar{h}\circ\ell_{\mathbf{P}_0}\circ\tilde{g} = h\circ\hat{g}' = \ell_{\mathbf{P}_0}\circ\tilde{g}, \quad h\circ\bar{h}\circ\ell_{\mathbf{P}_0}\circ\tilde{f} = h\circ\hat{f}' = \ell_{\mathbf{P}_0}\circ\tilde{f},$$

which imply, by the universal property of pushouts in TAlg$_\Sigma$, that $h\circ\bar{h} = \mathrm{Id}_{\overline{\mathbf{P}}_0}$. Therefore, $\bar{h}$ and h are isomorphisms inverse to each other.

This entails that we can take as the pushout of f and g in P-TAlg$_\Sigma$ the cocone

$$(\overline{\mathbf{P}}_0, \tilde{g}_0 : \mathbf{A}\to\overline{\mathbf{P}}_0, \tilde{f}_0 : \mathbf{B}\to\overline{\mathbf{P}}_0),$$

described above.

Let us prove now that the carrier P_0 of the partial Σ-algebra $\mathbf{P}_0$ is actually equal to the carrier $\overline{P}_0$ of its free completion $\overline{\mathbf{P}}_0$. This will imply that $\mathbf{P}_0 = \overline{\mathbf{P}}_0$, and in particular that $\mathbf{P}_0$ is a total algebra, and that $\ell_{\mathbf{P}_0} = \mathrm{Id}_{\mathbf{P}_0}$.

Let $A^{(0)}$, $B^{(0)}$, $P_0^{(0)}$ and $\overline{P}_0^{(0)}$ denote, respectively, the S-sets consisting of (the images of) the nullary operations defined in $\mathbf{A}$, $\mathbf{B}$, $\mathbf{P}_0$ and $\overline{\mathbf{P}}_0$. Let H_0 be the closed subset $C_{\overline{\mathbf{P}}_0}(\overline{P}_0^{(0)})$ of $\overline{\mathbf{P}}_0$ generated by $\overline{P}_0^{(0)}$. It turns out that $H_0\subseteq P_0$. Indeed, to begin with, notice that since, say, $\mathbf{A}'$ is a total Σ-algebra, all nullary operations are defined in it, which implies by construction that all nullary operations are defined in $\mathbf{P}_0$ and in particular $\overline{P}_0^{(0)} = P_0^{(0)} = \tilde{g}(A^{(0)})$. Then we have

$$\overline{P}_0^{(0)} = \tilde{g}(A^{(0)}) \subseteq \tilde{g}(C_{\mathbf{A}'}(A^{(0)})) \subseteq C_{\mathbf{P}_0}(P_0^{(0)}) \subseteq C_{\overline{\mathbf{P}}_0}(P_0^{(0)}),$$

where the second inclusion is a consequence of,[24] Proposition 3.6.2.(i). Now, since $\mathbf{A}'$ is a total algebra, its closed subalgebra supported on $C_{\mathbf{A}'}(A^{(0)})$ is also total, and then the relative subalgebra of $\overline{\mathbf{P}}_0$ supported on $\tilde{g}(C_{\mathbf{A}'}(A^{(0)}))$ is also total and thus closed. This shows that $\tilde{g}(C_{\mathbf{A}'}(A^{(0)}))$ is a closed subset of $\overline{\mathbf{P}}_0$ lying between $\overline{P}_0^{(0)}$ and $C_{\overline{\mathbf{P}}_0}(\overline{P}_0^{(0)})$, which implies, finally, that

$$H_0 = C_{\overline{\mathbf{P}}_0}(\overline{P}_0^{(0)}) = \tilde{g}(C_{\mathbf{A}'}(A^{(0)})) \subseteq P_0.$$

Let now $H = (\overline{P}_0 - P_0) \cup H_0$. For every $\varphi_0 \in \Omega^{(0)}$, $\varphi_0^{\overline{\mathbf{P}}_0} \in H$ because $\varphi_0^{\overline{\mathbf{P}}_0} \in H_0$. On the other hand, if $\varphi \in \Omega^{(+)}$ and $\underline{x} \in H^{\omega(\varphi)}$ were such that $\varphi^{\overline{\mathbf{P}}_0}(\underline{x}) \notin H$, then $\varphi^{\overline{\mathbf{P}}_0}(\underline{x}) \in P_0$, and hence, by Definition 5.3.1.(FC1) and Theorem 5.3.4 in,[24] we would have that $\underline{x} \in (H \cap P_0)^{\omega(\varphi)} = H_0^{\omega(\varphi)}$. But then, H_0 being closed in $\overline{\mathbf{P}}_0$, it would happen that $\varphi^{\overline{\mathbf{P}}_0}(\underline{x}) \in H_0 \subseteq H$, leading to a contradiction. Therefore, for every $\varphi \in \Omega$ and for every $\underline{x} \in H^{\omega(\varphi)}$, $\varphi^{\overline{\mathbf{P}}_0}(\underline{x}) \in H$ and hence H is a closed subset of $\overline{\mathbf{P}}_0$.

Let $\mathrm{Id}_H, \mathrm{Id}_{H_0} : \overline{\mathbf{P}}_0 \to \overline{\mathbf{P}}_0$ be the partial homomorphisms of total Σ-algebras defined by the set-theoretical identities on the closed subsets H and H_0 of $\overline{\mathbf{P}}_0$, respectively. Since $H \cap P_0 = H_0$, we have that

$$\mathrm{Id}_H \circ (\ell_{\mathbf{P}_0} \circ \tilde{g}) = \mathrm{Id}_{H_0} \circ (\ell_{\mathbf{P}_0} \circ \tilde{g}) \text{ and } \mathrm{Id}_H \circ (\ell_{\mathbf{P}_0} \circ \tilde{f}) = \mathrm{Id}_{H_0} \circ (\ell_{\mathbf{P}_0} \circ \tilde{f}).$$

By the universal property of pushouts in P-TAlg$_\Sigma$, this implies that $H = H_0$ and hence $P_0 = \overline{P}_0$, as we wanted to prove.

Thus, $\mathbf{P}_0 = \overline{\mathbf{P}}_0$ and $\ell_{\mathbf{P}_0} = \mathrm{Id}_{\mathbf{P}_0}$, and therefore we can take as the pushout in P-TAlg$_\Sigma$ of f and g the cocone

$$(\mathbf{P}_0, \tilde{g}_0 : \mathbf{A} \to \mathbf{P}_0, \tilde{f}_0 : \mathbf{B} \to \mathbf{P}_0),$$

where, now, $\tilde{g}_0 : \mathbf{A} \to \mathbf{P}_0$ and $\tilde{f}_0 : \mathbf{B} \to \mathbf{P}_0$ are the partial homomorphisms defined by the homomorphisms $\tilde{g} : \mathbf{A}' \to \mathbf{P}_0$ and $\tilde{f} : \mathbf{B}' \to \mathbf{P}_0$ in the pushout $(\mathbf{P}_0, \tilde{g} : \mathbf{A}' \to \mathbf{P}_0, \tilde{f} : \mathbf{B}' \to \mathbf{P}_0)$ of $f' : \mathbf{K}' \to \mathbf{A}'$ and $g' : \mathbf{K}' \to \mathbf{B}'$ in Alg$_\Sigma$. Since, as we have just proved, $\mathbf{P}_0$ is a total Σ-algebra, if we prove that this cocone is also a pushout of f and g in CDQ-Alg$_\Sigma$, we will be done.

So, let $p : \mathbf{A} \to \mathbf{C}$ and $q : \mathbf{B} \to \mathbf{C}$ be two cd-quomorphisms of partial Σ-algebras such that $p \circ g = q \circ f$, and let $\ell_{\mathbf{C}} : \mathbf{C} \to \overline{\mathbf{C}}$ be a free completion of $\mathbf{C}$: as always, we assume without any loss of generality that this is an embedding of a relative subalgebra. By the universal property of pushouts in P-TAlg$_\Sigma$, there exists one, and only one, partial homomorphism $h : \mathbf{P}_0 \to \overline{\mathbf{C}}$ such that $h \circ \tilde{g}_0 = \ell_{\mathbf{C}} \circ p$ and $h \circ \tilde{f}_0 = \ell_{\mathbf{C}} \circ q$.

$$\begin{tikzcd}
\mathbf{K} \arrow[r, "f"] \arrow[d, "g"'] & \mathbf{A} \arrow[d, "\tilde{g}_0"] \arrow[ddr, bend left, "p"] \arrow[dddrr, bend left=40, "\ell_{\mathbf{C}} \circ p"] & & \\
\mathbf{B} \arrow[r, "\tilde{f}_0"'] \arrow[drr, bend right, "q"'] \arrow[ddrrr, bend right=40, "\ell_{\mathbf{C}} \circ q"'] & \mathbf{P}_0 \arrow[dr, dotted, "h"] \arrow[ddrr, bend left, "h"] & & \\
 & & \mathbf{C} \arrow[dr, hook] & \\
 & & & \overline{\mathbf{C}}
\end{tikzcd}$$

Since, by construction, $P_0 = \tilde{g}_0(A') \cup \tilde{f}_0(B')$, we have that $h(P_0) \subseteq C$, and since $\mathbf{P}_0$ is total, the relative subalgebra of $\mathbf{C}$ supported on $h(P_0)$ is also total. This implies that the partial homomorphism $h : \mathbf{P}_0 \to \overline{\mathbf{C}}$ is actually a cd-quomorphism $h : \mathbf{P}_0 \to \mathbf{C}$. And it is the only cd-quomorphism $\mathbf{P}_0 \to \mathbf{C}$ such that $h \circ \tilde{g}_0 = p$ and $h \circ \tilde{f}_0 = q$. Indeed, if $\bar{h} : \mathbf{P}_0 \to \mathbf{C}$ is another cd-quomorphism such that $\bar{h} \circ \tilde{g}_0 = p$ and $\bar{h} \circ \tilde{f}_0 = q$, then $\ell_{\mathbf{C}} \circ \bar{h} : \mathbf{P}_0 \to \overline{\mathbf{C}}$ is a cd-quomorphism, i.e., a partial homomorphism of total algebras, such that $\ell_{\mathbf{C}} \circ \bar{h} \circ \tilde{g}_0 = \ell_{\mathbf{C}} \circ p$ and $\ell_{\mathbf{C}} \circ \bar{h} \circ \tilde{f}_0 = \ell_{\mathbf{C}} \circ q$, which implies that $\ell_{\mathbf{C}} \circ \bar{h} = h : \mathbf{P}_0 \to \overline{\mathbf{C}}$ and hence, $\ell_{\mathbf{C}}$ being injective, that $\bar{h} = h : \mathbf{P}_0 \to \mathbf{C}$.

This finishes the proof of the fact that the pushout of f and g in P-TAlg$_\Sigma$ given above is also their pushout in CDQ-Alg$_\Sigma$. ■

Using Theorems 1 and 3, we can rephrase this result in an operational way.

Corollary 1 *Two partial homomorphisms of total Σ-algebras $f : \mathbf{K} \to \mathbf{A}$ and $g : \mathbf{K} \to \mathbf{B}$ have a pushout in* P-TAlg$_\Sigma$ *if and only if they satisfy the following three conditions:*

PHPO1) *There exist the greatest closed subsets A' and B' of $\mathbf{A}$ and $\mathbf{B}$, respectively, such that $f^{-1}(A') = g^{-1}(B')$.*

Let $K' = f^{-1}(A') = g^{-1}(B')$, let $\mathbf{K}'$, $\mathbf{A}'$ and $\mathbf{B}'$ be the subalgebras of $\mathbf{K}$, $\mathbf{A}$ and $\mathbf{B}$ supported on their closed subsets K', A' and B', respectively, and let $f' : \mathbf{K}' \to \mathbf{A}'$ and $g' : \mathbf{K}' \to \mathbf{B}'$ be the homomorphisms between these algebras induced by the partial homomorphisms f and g. Let finally θ be the congruence on the disjoint sum $\mathbf{A}'^{(+)} \oplus \mathbf{B}'^{(+)}$ generated by the relation

$$E(f', g') = \Big(\{(f_s(k), g_s(k)) \mid k \in K'_s\}\Big)_{s \in S}.$$

PHPO2) *For every $\varphi \in \Omega^{(+)}$,*

$$((A' \sqcup B')/\theta)^{\omega(\varphi)} = nat_\theta(A')^{\omega(\varphi)} \cup nat_\theta(B')^{\omega(\varphi)}.$$

PHPO3) *For every closed subsets A_0 and B_0 of $\mathbf{A}'$ and $\mathbf{B}'$, respectively, such that $f^{-1}(A_0) = g^{-1}(B_0)$, if θ_0 is the congruence on $\mathbf{A}_0^{(+)} \oplus \mathbf{B}_0^{(+)}$ generated by $E(f', g') \cap (A_0 \sqcup B_0)^2$, then*

$$\theta \cap ((A_0 \sqcup B_0) \times (A' \sqcup B')) \subseteq \theta_0.$$

And if f and g satisfy these conditions, then a pushout of them in P-TAlg$_\Sigma$ *is given by the cocone*

$$(\mathbf{P}, \tilde{g} : \mathbf{A} \to \mathbf{P}, \tilde{f} : \mathbf{B} \to \mathbf{P})$$

where:

- $\mathbf{P}$ *is the total Σ-algebra whose $\Sigma^{(+)}$-reduct is $(\mathbf{A}'^{(+)} \oplus \mathbf{B}'^{(+)})/\theta$ and where $\varphi_0^{\mathbf{P}} = [\varphi_0^{\mathbf{A}}]_\theta = [\varphi_0^{\mathbf{B}}]_\theta$ for every $\varphi_0 \in \Omega^{(0)}$;*
- $\tilde{f} : \mathbf{B} \to \mathbf{P}$ *and $\tilde{g} : \mathbf{A} \to \mathbf{P}$ are the partial homomorphisms defined by the restrictions to B' and A' of the quotient mapping $nat_\theta : A' \sqcup B' \to (A' \sqcup B')/\theta$.*

Proof. Notice that, since $\mathbf{A}$ and $\mathbf{B}$ are total algebras, condition (PHPO1) is equivalent to condition (CDCPO1) in Theorem 3, while conditions (PHPO2) and (PHPO3) are exactly conditions (CDCPO2) and (CDCPO3) therein. Then, the equivalence between (ii) and (iii) in Theorem 5 applies. On the other hand, the description of a pushout in P-TAlg$_\Sigma$ is also a translation of the description of a pushout in CDCQ-Alg$_\Sigma$ given in Theorem 3, using the description of a pushout in Alg$_\Sigma$ given in §2.2 and the fact that, since $\mathbf{K}'$, $\mathbf{A}'$ and $\mathbf{B}'$ are total algebras, $(\varphi_0^{\mathbf{A}}, \varphi_0^{\mathbf{B}}) = (f(\varphi_0^{\mathbf{K}}), g(\varphi_0^{\mathbf{K}})) \in E(f', g')$. ■

Remark 1 *R. Alberich and F. Rosselló*[12] *proved that, for* one-sorted signatures with an operation symbol at least binary, *condition (CHPO2) is equivalent to the fact that* $(A' \sqcup B')/\theta = nat_\theta(A')$ *or* $(A' \sqcup B')/\theta = nat_\theta(B')$*: see Corollary 4.2 in the cited paper.*

Remark 2 *Although in Theorem 5 and Corollary 1 we only considered pushouts of* pairs *of partial homomorphisms, these results generalize in a straightforward way to pushouts of arbitrary non-empty families of partial homomorphisms of total algebras with the same source algebra.*

Conditions (PHPO1), (PHPO2) and (PHPO3) in Corollary 1 are independent of each other, as the following examples show.

Example 1 *Let* Σ *be a one-sorted signature with only one operation symbol* φ*, which is binary. Let* $\mathbf{K}$*,* $\mathbf{A}$ *and* $\mathbf{B}$ *be the total* Σ*-algebras with carriers* $K = \{k_1, k_2\}$*,* $A = \{a_1, a_2, a_3\}$ *and* $B = \{b\}$*, respectively, and their operations defined as follows:*

- $\varphi^{\mathbf{K}}(k_i, k_i) = k_i$ *for every* $i = 1, 2$ *and* $\varphi^{\mathbf{K}}(k_1, k_2) = \varphi^{\mathbf{K}}(k_2, k_1) = k_2$*,*
- $\varphi^{\mathbf{A}}(a_i, a_i) = a_i$ *for every* $i = 1, 2, 3$ *and* $\varphi^{\mathbf{A}}(a_i, a_j) = a_2$ *for every* $i, j = 1, 2, 3$ *with* $i \neq j$*,*
- $\varphi^{\mathbf{B}}(b, b) = b$*.*

Let $f : \mathbf{K} \to \mathbf{A}$ *and* $g : \mathbf{K} \to \mathbf{B}$ *be the partial homomorphisms defined by* $f(k_1) = a_1$ *and* $f(k_2) = a_2$*, and* $g(k_1) = b$*, respectively. These partial homomorphisms do not satisfy condition (PHPO1), because* $A_1 = \{a_1\}$ *and* $B_1 = B$ *are two closed subsets of* $\mathbf{A}$ *and* $\mathbf{B}$ *with the same preimage in* $\mathbf{K}$*, as well as* $A_2 = \{a_3\}$ *and* $B_2 = \emptyset$*, and the only closed subset of* $\mathbf{A}$ *containing* A_1 *and* A_2 *is its carrier* A*, but there does not exist any closed subset* B' *of* $\mathbf{B}$ *such that* $f^{-1}(A) = g^{-1}(B')$*.*

Example 2 *Let* Σ *be the same signature as in the last example, and let now* $\mathbf{K}$*,* $\mathbf{A}$ *and* $\mathbf{B}$ *be the total* Σ*-algebras with carriers* $K = \{k\}$*,* $A = \{a_1, a_2\}$ *and* $B = \{b_1, b_2\}$*, respectively, and their operations defined as follows:*

- $\varphi^{\mathbf{B}}(k, k) = k$*,*
- $\varphi^{\mathbf{A}}(a_1, a_1) = a_1$ *and* $\varphi^{\mathbf{A}}(a_1, a_2) = \varphi^{\mathbf{A}}(a_2, a_1) = \varphi^{\mathbf{A}}(a_2, a_2) = a_2$*,*
- $\varphi^{\mathbf{B}}(b_1, b_1) = b_1$ *and* $\varphi^{\mathbf{B}}(b_1, b_2) = \varphi^{\mathbf{B}}(b_2, b_1) = \varphi^{\mathbf{B}}(b_2, b_2) = b_2$*.*

Let $f : \mathbf{K} \to \mathbf{A}$ *and* $g : \mathbf{K} \to \mathbf{B}$ *be the homomorphisms defined by* $f(k) = a_1$ *and* $g(k) = b_1$*. If we consider them as partial homomorphisms, it is clear that they satisfy condition (PHPO1), with* $A' = A$ *and* $B' = B$*. Now, the congruence* θ *on* $\mathbf{A} \oplus \mathbf{B}$ *generated by* $E(f, g)$ *only identifies* a_1 *with* b_1*, and it is straightforward to prove that* f *and* g *satisfy condition (PHPO3). But they do not satisfy condition (PHPO2):* $(A \sqcup B)/\theta = \{[a_1]_\theta = [b_1]_\theta, [a_2]_\theta, [b_2]_\theta\}$ *and then*

$$((A \sqcup B)/\theta)^2 \neq \{[a_1]_\theta, [a_2]_\theta\}^2 \cup \{[b_1]_\theta, [b_2]_\theta\}^2.$$

Example 3 *Let be again* Σ *the same signature as in the last two examples, and let now* $\mathbf{K}$*,* $\mathbf{A}$ *and* $\mathbf{B}$ *be the total* Σ*-algebras with carriers* $K = \{k_1, k_2\}$*,* $A = \{a\}$ *and* $B = \{b_1, b_2, c, d_1, d_2\}$*, respectively, and their operations defined as follows:*

- $\varphi^{\mathbf{K}}(k_1, k_1) = k_1$ *and* $\varphi^{\mathbf{K}}(k_1, k_2) = \varphi^{\mathbf{K}}(k_2, k_1) = \varphi^{\mathbf{K}}(k_2, k_2) = k_2$*,*
- $\varphi^{\mathbf{A}}(a, a) = a$*,*

- *$\varphi^{\mathbf{B}}(t,t) = t$ for every $t \in B$, $\varphi^{\mathbf{B}}(b_1,b_2) = \varphi^{\mathbf{B}}(b_2,b_1) = b_2$, $\varphi^{\mathbf{B}}(b_i,c) = \varphi^{\mathbf{B}}(c,b_i) = d_i$ for every $i = 1,2$, $\varphi^{\mathbf{B}}(b_i,d_j) = \varphi^{\mathbf{B}}(d_j,b_i) = d_j$ for every $i,j = 1,2$, $\varphi^{\mathbf{B}}(c,d_j) = \varphi^{\mathbf{B}}(d_j,c) = d_j$ for every $j = 1,2$, and $\varphi^{\mathbf{B}}(d_1,d_2) = \varphi^{\mathbf{B}}(d_2,d_1) = d_2$.*

Let $f : \mathbf{K} \to \mathbf{A}$ and $g : \mathbf{K} \to \mathbf{B}$ be the homomorphisms defined by $f(k_1) = f(k_2) = a$ and $g(k_1) = b_1$, $g(k_2) = b_2$, respectively. As before, these homomorphisms satisfy condition (PHPO1) with $A' = A$ and $B' = B$, and in this case the congruence θ on $\mathbf{A} \oplus \mathbf{B}$ generated by $E(f,g)$ identifies on the one hand a, b_1 and b_2, and on the other hand d_1 and d_2. It is clear then that f and g satisfy condition (PHPO2): actually, the restriction of $nat_\theta : A \sqcup B \to (A \sqcup B)/\theta$ to B is surjective. But they do not satisfy condition (PHPO3): $A_0 = \emptyset$ and $B_0 = \{d_1,d_2\}$ are closed subsets of $\mathbf{A}$ and $\mathbf{B}$ such that $f^{-1}(A_0) = g^{-1}(B_0)$ and the congruence on their disjoint sum generated by the restriction of $E(f,g)$ is clearly the identity relation, while $\theta \cap ((A_0 \sqcup B_0) \times (A \sqcup B))$ identifies d_1 and d_2.

Condition (PHPO3) is quite involved. It was shown in Theorem 6 and Proposition 7 of[11] that, for quomorphisms, a similar condition could be replaced by a simpler one, and one could wonder whether it would also happen in the partial homomorphisms setting. So, let us introduce the condition for partial homomorphisms that would correspond to that simpler condition for quomorphisms and check whether it can replace (PHPO3).

Let $f : \mathbf{K} \to \mathbf{A}$ and $g : \mathbf{K} \to \mathbf{B}$ be two partial homomorphisms of total Σ-algebras satisfying condition (PHPO1). As in Corollary 1, let A' and B' be the greatest closed subsets of $\mathbf{A}$ and $\mathbf{B}$, respectively, such that $f^{-1}(A') = g^{-1}(B')$, let K' denote this subset of $\mathbf{K}$, let $\mathbf{K}'$, $\mathbf{A}'$ and $\mathbf{B}'$ be the subalgebras of $\mathbf{K}$, $\mathbf{A}$ and $\mathbf{B}$ supported on their closed subsets K', A' and B', respectively, let $f' : \mathbf{K}' \to \mathbf{A}'$ and $g' : \mathbf{K}' \to \mathbf{B}'$ be the homomorphisms between these algebras induced by the partial homomorphisms f and g, and let

$$E(f',g') = \Big(\{(f_s(k), g_s(k)) \mid k \in K'_s\}\Big)_{s\in S}.$$

Consider the following condition on f and g.

PHPO4) The equivalence relation θ_e on $A' \sqcup B'$ generated by $E(f',g')$ is a congruence on $\mathbf{A}'^{(+)} \oplus \mathbf{B}'^{(+)}$ (and therefore it is equal to the congruence θ introduced in Corollary 1).

Proposition 2 *Let $f : \mathbf{K} \to \mathbf{A}$ and $g : \mathbf{K} \to \mathbf{B}$ be two partial homomorphisms of total Σ-algebras satisfying condition (PHPO1). If they satisfy condition (PHPO4), then they satisfy condition (PHPO3).*

Proof. Assume that f and g satisfy conditions (PHPO1) and (PHPO4). Let A' and B' be the closed subsets of $\mathbf{A}$ and $\mathbf{B}$ given by condition (PHPO1), let $\mathbf{A}_0$ and $\mathbf{B}_0$ be closed subalgebras of $\mathbf{A}'$ and $\mathbf{B}'$, supported respectively on A_0 and B_0, such that $f^{-1}(A_0) = g^{-1}(B_0)$, and let θ_0 be the congruence on $\mathbf{A}_0^{(+)} \oplus \mathbf{B}_0^{(+)}$ generated by $E(f',g') \cap (A_0 \sqcup B_0)^2$.

Let (d,d') be an element of $\theta \cap ((A_0 \sqcup B_0) \times (A' \sqcup B'))$; since, by assumption, $\theta = \theta_e$, we have that

$$(d,d') \in \theta_e \cap ((A_0 \sqcup B_0) \times (A' \sqcup B'))$$

and thus $d \in A_0 \sqcup B_0$ and there exist some

$$(d,d_1),(d_1,d_2),\ldots,(d_{n-1},d') \in E(f',g') \cap (A_0 \sqcup B_0)^2.$$

Now, the equality $f^{-1}(A_0) = g^{-1}(B_0)$ implies that

$$E(f', g') \cap ((A_0 \sqcup B_0) \times (A' \sqcup B')) \subseteq \theta_0,$$

and then, since $d \in D_0$, we have that

$$(d, d_1), (d_1, d_2), \ldots, (d_{n-1}, d') \in \theta_0.$$

Therefore $(d, d') \in \theta_0$. This proves that $\theta \cap ((A_0 \sqcup B_0) \times (A' \sqcup B')) \subseteq \theta_0$, as required by condition (PHPO3). ■

Unfortunately, the converse implication is false, even for partial homomorphisms satisfying also condition (PHPO2), as the following example shows. Therefore, although (PHPO4), together with (PHPO1) and (PHPO2), provides a sufficient condition for the existence of a pushout in P-TAlg$_\Sigma$ of two partial homomorphisms of total algebras, it is not necessary, and therefore it cannot replace (PHPO3).

Example 4 *Let Σ be a one-sorted signature with only a binary operation symbol φ and two unary operation symbols ψ and ψ_1. Let $\mathbf{K}$, $\mathbf{A}$ and $\mathbf{B}$ be the total Σ-algebras with carriers $\mathbf{K} = \{k_1, k_2\}$, $\mathbf{A} = \{a\}$ and $\mathbf{B} = \{b_1, b_2, c, d_1, d_2\}$, respectively, and their operations defined by:*

- $\varphi^{\mathbf{K}}(k_1, k_2) = \varphi^{\mathbf{K}}(k_2, k_1) = k_2$, $\varphi^{\mathbf{K}}(k_i, k_i) = \psi^{\mathbf{K}}(k_i) = \psi_1^{\mathbf{K}}(k_i) = k_i$ *for every* $i = 1, 2$.
- $\varphi^{\mathbf{A}}(a, a) = \psi^{\mathbf{A}}(a) = \psi_1^{\mathbf{A}}(a) = a$.
- $\varphi^{\mathbf{B}}$ *is defined by* $\varphi^{\mathbf{B}}(b_1, b_2) = \varphi^{\mathbf{B}}(b_2, b_1) = b_2$, $\varphi^{\mathbf{B}}(b_i, b_i) = b_i$ *for every* $i = 1, 2$, $\varphi^{\mathbf{B}}(c, c) = c$, $\varphi^{\mathbf{B}}(b_i, c) = \varphi^{\mathbf{B}}(c, b_i) = \varphi^{\mathbf{B}}(d_i, c) = \varphi^{\mathbf{B}}(c, d_i) = d_i$ *for every* $i = 1, 2$, $\varphi^{\mathbf{B}}(d_i, d_j) = c$ *for every* $i, j = 1, 2$, *and* $\varphi^{\mathbf{B}}(b_i, d_j) = \varphi^{\mathbf{B}}(d_j, b_i) = d_j$ *for every* $i, j = 1, 2$*; the unary operations on* $\mathbf{B}$ *are defined by* $\psi^{\mathbf{B}}(d_1) = d_2$, $\psi^{\mathbf{B}}(d_2) = d_1$, $\psi_1^{\mathbf{B}}(d_1) = b_1$, $\psi_1^{\mathbf{B}}(d_2) = b_2$, $\psi^{\mathbf{B}}(c) = d_1$, $\psi_1^{\mathbf{B}}(c) = d_2$, *and* $\psi_1^{\mathbf{B}}(b_i) = \psi^{\mathbf{B}}(b_i) = b_i$ *for every* $i = 1, 2$.

Let $f : \mathbf{K} \to \mathbf{A}$ and $g : \mathbf{K} \to \mathbf{B}$ be the homomorphisms defined by $f(k_1) = f(k_2) = a$ and $f(k_1) = b_1$, $f(k_2) = b_2$.

Since f and g are total, they satisfy condition (PHPO1) with $A' = A$ and $B' = B$. The equivalence θ_e on $A \sqcup B$ generated by $E(f, g)$ identifies b_1, b_2 and a, while the congruence θ on $\mathbf{A} \oplus \mathbf{B}$ generated by $E(f, g)$ identifies moreover d_1 and d_2. Therefore, $\theta_e \neq \theta$ and hence f and g do not satisfy condition (PHPO4). In particular, by Corollary 5.2 in,[12] they do not have a pushout in CQ-Alg$_\Sigma$.

Now, it is clear that f and g satisfy condition (PHPO2): the restriction of $nat_\theta : A \sqcup B \to (A \sqcup B)/\theta$ to B is surjective. And they turn out to satisfy also condition (PHPO3). Indeed, it is easy to check that the only pairs of closed subsets of $\mathbf{A}$ and $\mathbf{B}$ with the same preimage in $\mathbf{K}$ are

$$f(K), g(K) \quad \text{and} \quad A, B,$$

and that on these pairs of sets the inclusion of relations required by condition (PHPO3) is satisfied. Therefore, f and g have a pushout in P-TAlg$_\Sigma$*, and it is given by their pushout in* Alg$_\Sigma$.

Let us point out that an argument similar to the one used in the proof of the implications (i)⟹(ii) and (ii)⟹(iii) in Theorem 5 proves also the following result. We leave the details to the reader.

Corollary 2 *Let $f : \mathbf{K} \to \mathbf{A}$ and $g : \mathbf{K} \to \mathbf{B}$ be two partial homomorphisms of total Σ-algebras. If f and g have a pushout in* CQ-Alg$_\Sigma$ *and it is given by a total Σ-algebra, then they have a pushout in* P-TAlg$_\Sigma$ *and it equal to their pushout in* CQ-Alg$_\Sigma$. ■

Example 4 shows that the converse implication in this proposition is false: two partial homomorphisms of total Σ-algebras having a pushout in P-TAlg$_\Sigma$ need not have a pushout in CQ-Alg$_\Sigma$. On the other hand, it is not difficult to produce (similar) examples showing that:

- two partial homomorphisms of total Σ-algebras having a pushout in Q-Alg$_\Sigma$ given by a total Σ-algebra need not have a pushout in P-TAlg$_\Sigma$;
- two partial homomorphisms of total Σ-algebras having a pushout in P-TAlg$_\Sigma$ can have a pushout in CQ-Alg$_\Sigma$ given by a non-total Σ-algebra.

The interested reader can find such examples in,[25] Chapter 8.

Furthermore, and against what happens in the unary case,[3] the SPO transformation in the category P-TAlg$_\Sigma$ for an arbitrary signature Σ is independent of the double-pushout transformation of total Σ-algebras in TAlg$_\Sigma$ as introduced by P. Burmeister, M. Llabrés and F. Rosselló,[10] in the sense that, if we translate a partial homomorphism of total Σ-algebras $r : \mathbf{L} \to \mathbf{R}$ into the DPO production rule $P_r = (\mathbf{L} \hookleftarrow \mathbf{Dom}\, r \xrightarrow{r} \mathbf{R})$, then there exist homomorphisms $f : \mathbf{L} \to \mathbf{G}$ of total Σ-algebras such that:

- r and f have a pushout in P-TAlg$_\Sigma$ but P_r cannot be applied to $\mathbf{G}$ through f in TAlg$_\Sigma$;
- P_r can be applied to $\mathbf{G}$ through f in TAlg$_\Sigma$ but r and f do not have a pushout in P-TAlg$_\Sigma$;
- P_r can be applied to $\mathbf{G}$ through f in TAlg$_\Sigma$ *and* r and f have a pushout in P-TAlg$_\Sigma$, but the corresponding derived algebras are different.

The interested reader can find the corresponding examples in,[26] Chapter 5.

To finish this paper, let us give an easy example of SPO transformation of total algebras.

Example 5 *Let Σ_G be the one-sorted signature of groups, with a nullary operation e (corresponding to the neutral element), a unary operation $^{-1}$ (corresponding to the inversion), and a binary operation $*$. Let $\mathbf{F}(x)$ be the free group generated by one element x, let $\mathbf{E}$ be a trivial group, with only one element, and let $r : \mathbf{F}(x) \to \mathbf{E}$ be the partial homomorphism defined simply by $r(e^{\mathbf{F}(x)}) = e^{\mathbf{E}}$.*

Let now $\mathbf{G}$ be any group and $f : \mathbf{F}(x) \to \mathbf{G}$ any homomorphism of groups, which is uniquely determined by the image $f(x)$. Then, we can understand r as an SPO transformation rule that can be applied to $\mathbf{G}$ through the occurrence f when they have a pushout in P-TAlg$_{\Sigma_G}$. Corollary 1 allows us to decide when such an application is possible and what is its result.

It turns out that f and r satisfy condition (PHPO1) if and only if there exists the greatest subgroup $\mathbf{H}$ of $\mathbf{G}$ whose intersection with the subgroup $\langle f(x)\rangle$ generated by $f(x)$ is the neutral element of $\mathbf{G}$, in which case the closed subsets of $\mathbf{E}$ and $\mathbf{G}$ asked by this condition are (the carriers of) $\mathbf{E}$ itself and $\mathbf{H}$. Now, the congruence θ on $\mathbf{E}^{(+)} \oplus \mathbf{H}^{(+)}$ defined in Corollary 1 simply identifies the neutral elements of $\mathbf{E}$ and $\mathbf{H}$ and $(\mathbf{E}^{(+)} \oplus \mathbf{H}^{(+)})/\theta \cong \mathbf{H}^{(+)}$. It is straightforward to check then that f and r satisfy conditions (PHPO2) and (PHPO3).

Summarizing, r can be applied to $\mathbf{G}$ through f if and only if there exists the greatest subgroup $\mathbf{H}$ of $\mathbf{G}$ such that $\mathbf{H} \cap \langle f(x)\rangle = \{e^{\mathbf{G}}\}$, and when it can be applied, the derived Σ_G-algebra is, up to isomorphism, this subgroup $\mathbf{H}$.

For instance, if $\mathbf{G}$ is a permutation group generated by several pairwise disjoint transpositions $\sigma_1, \ldots, \sigma_k$, and if the homomorphism $f : \mathbf{F}(x) \to \mathbf{G}$ is defined by $f(x) = \sigma_1$, then r can be applied to $\mathbf{G}$ through it and the result of this application is the subgroup $\langle \sigma_2, \ldots, \sigma_k\rangle$ of $\mathbf{G}$ generated by the other transpositions. This kind *of permutation groups are used by C. Reidys and P. Stadler*[27] *as algebraic models of RNA secondary structures, and removing one of these generators corresponds to the biochemical operation of breaking a bond in the structure without removing any nucleotide.*

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