

On the coverings by tolerance classes¹

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Abstract

A tolerance is a reflexive and symmetric, but not necessarily transitive, binary relation. Contrary to what happens with equivalence relations, when dealing with tolerances one must distinguish between *blocks* (maximal subsets where the tolerance is a total relation) and *classes* (the class of an element is the set of those elements tolerable with it). Both blocks and classes of a tolerance on a set define coverings of this set, but not every covering of a set is defined in this way. The characterization of those coverings that are families of blocks of some tolerance has been known for more than a decade now. In this paper we give a characterization of those coverings of a finite set that are families of classes of some tolerance.

Key words: Tolerance; Similarity relation; Class; Neighborhood; Block

1 Introduction

A *tolerance* is a reflexive and symmetric, but not necessarily transitive, binary relation on a set. Tolerances provide a general tool for studying indiscernibility phenomena, and with this purpose they were already introduced by Poincaré [6], Carnap [1], who called them similarities, and, seemingly independently, by Zeeman [13], to whom we owe the term tolerance. Tolerances have recently found relevant applications in rough set theory [3,5,8,11], digital mathematics [4,12], and other areas, and their broad usefulness makes them ubiquitous in mathematics and computer science, often under other names, depending

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on the field where they appear: compatibility relations, similarity relations, proximity relations. We can even identify a tolerance T with an undirected graph. The nodes of the graph would constitute the set on which the tolerance is formed, and two elements are related if, and only if, there is an arc in the graph connecting them. Every tolerance defines such a graph, and the reflexive closure of the adjacency relation on an undirected graph is a tolerance. Actually, in the digital topology literature undirected graphs without self-loops are often called *tolerance graphs* [9]. Schroeder and Wright [10] offer a detailed survey of the theory of tolerances and their applications.

A special, and very well-known, type of tolerances are the *equivalence relations*, which are nothing but the transitive tolerances. Recall that an equivalence relation θ on a set X defines a partition of this set, given by the equivalence classes modulo θ , and that this characterizes those coverings of a set that are families of classes of equivalence relations.

Arbitrary tolerances are more complicated in this respect. For one thing, the notion of equivalence class splits into two different concepts: the *blocks* and the *classes* of the tolerance. Given a tolerance T on a set X , a T -*block* is a subset of X such that the restriction of T to this subset is a total relation, and maximal with respect to this property. On the other hand, the T -*class* of an element $x \in X$ is the subset of those elements of X that are tolerant with x . If we understand a tolerance as an undirected graph, its blocks correspond to its maximal cliques and the class of a node corresponds to its set of adjacent nodes, together with the node itself. Tolerance classes are also called T -*neighborhoods* and *tolerance sets* in the literature.

The blocks and the classes of a tolerance on a set X form coverings of X , but these coverings need not be partitions. Actually, if one of these coverings is a partition of X , then the tolerance is an equivalence relation and the coverings by blocks and by classes are exactly the same. Coverings by blocks and by classes have very different nature and properties. For instance, the covering by blocks of a tolerance determines completely the tolerance, because a pair of elements belongs to the tolerance if and only if both elements belong simultaneously to some block. This eases the task of characterizing those coverings of a set that are families of blocks of tolerances. Such a characterization can be found, for instance, in Corollary 4.25 of Schroeder and Wright's survey; in Section 2 below we recall and slightly simplify this result.

Instead, different tolerances may yield the same covering by classes. For instance, we invite the reader to check that the covering of $\{1, 2, 3, 4\}$ formed by the subsets

$$\{2, 3, 4\}, \{1, 3, 4\}, \{1, 2, 4\}, \{1, 2, 3\}$$

can be obtained as the family of classes of three different tolerances on this set (we present them in Example 6). This lack of bijective correspondence

between tolerances and their families of classes not only makes it difficult to characterize those coverings of a set that are families of tolerance classes, it also has incidence in the practical application of tolerances. For instance, coverings of finite sets by tolerance classes without the explicit knowledge of the class of each element are often used in rough set theory. Then, in these applications, two tolerances with the same coverings by classes yield the same constructions: Cattaneo's upper and lower approximations of sets [2], Polkowski and Skowron's indiscernibility relations [7].

We have been working on the problem of the characterization of coverings by classes, and in this paper we answer the following two questions: given a covering of a *finite set* X , when is it the family of classes of some tolerance on X ? and, when is it the family of classes of more than one tolerance on X ? Or in graph theoretical terms: given a covering of a finite set X , when does there exist an undirected graph G (without self-loops) with set of nodes X whose adjacency sets are exactly the members of this family? and, when do more than one such graph exist?

Unfortunately, our proofs for finite sets cannot be translated in a simple way to arbitrary sets, as they rely strongly at several places on the fact that an inclusion of finite sets of the same cardinality is always an equality. We believe that suitable answers to these questions in the arbitrary case will come from using a different approach. Anyway, most applications of tolerances in rough set theory and digital mathematics are based on finite sets, and therefore we consider this case interesting in itself.

2 Blocks and classes of a tolerance

Let T be a *tolerance* on a set X , i.e., a reflexive and symmetric binary relation on X . A T -preblock is a subset Y of X such that $Y \times Y \subseteq T$. A T -block is a maximal T -preblock, i.e., a T -preblock that is not strictly contained in any other T -preblock.

Lemma 1. *Let T be a tolerance on a set X . A subset Y of X is a T -preblock if and only if it is contained in some T -block.*

Proof. It is straightforward that every subset of a T -block is a T -preblock. Conversely, every T -preblock is contained in a T -block: when the set X is finite it is an obvious consequence of the existence of maximal elements in every finite non-empty poset, and in the general case it is a consequence of Zorn's Lemma, since the union of an ascending chain of T -preblocks is again a T -preblock. \square

Let $\mathcal{B}(T)$ denote the family of all T -blocks. Since every singleton is a T -preblock, the previous lemma entails that every element of X is contained in some T -block and therefore $\mathcal{B}(T)$ is a covering of X . Another consequence of Lemma 1 is that a tolerance T on a set X is completely determined by its blocks: namely,

$$(x, y) \in T \iff \text{there exists some } B \in \mathcal{B}(T) \text{ such that } x, y \in B,$$

because $\{x, y\}$ is a T -preblock if and only if $(x, y) \in T$. In particular, if T and T' are tolerances on a set X such that $\mathcal{B}(T) = \mathcal{B}(T')$, then $T = T'$.

Not every covering of a set X is of the form $\mathcal{B}(T)$ for some tolerance T . The following result, which slightly generalizes [10, Cor. 4.25], characterizes those coverings that are of this form. This will be one of the key ingredients in the proof of the characterization of the coverings of finite sets by tolerance classes we establish in Section 4.

Theorem 2. *Let \mathcal{B} be a covering of a set X . Then $\mathcal{B} = \mathcal{B}(T)$ for some tolerance T on X if and only if \mathcal{B} satisfies the following two conditions:*

(i) *For every $B, B' \in \mathcal{B}$, if $B \subseteq B'$ then $B = B'$.*

(ii) *For every $Y \subseteq X$, if each pair of points in Y belong to some member of \mathcal{B} , then Y is contained in some member of \mathcal{B} .*

Proof. Let T be a tolerance on X such that $\mathcal{B} = \mathcal{B}(T)$. Then, condition (i) for \mathcal{B} is straightforward, by the maximality of T -blocks. As far as condition (ii) goes, if $Y \subseteq X$ is such that every pair of points in it belong to some element of \mathcal{B} , then Y is a T -preblock and therefore, by Lemma 1, it is contained in some T -block, i.e., in some element of \mathcal{B} .

Conversely, let \mathcal{B} be a covering of X satisfying conditions (i) and (ii), and let

$$T = \bigcup_{B \in \mathcal{B}} B \times B = \{(x, y) \in X \times X \mid \text{there exists } B \in \mathcal{B} \text{ such that } x, y \in B\}.$$

This relation is clearly a tolerance such that every $B \in \mathcal{B}$ is a T -preblock, and condition (ii) says that every T -preblock is contained in some element B of \mathcal{B} . This implies that every T -block belongs to \mathcal{B} and, together with (i), that every element of \mathcal{B} is a T -block. \square

Let T be again a tolerance on a set X . The T -class of an element x of X is the set

$$T(x) = \{y \in X \mid (x, y) \in T\}.$$

Let $\mathcal{K}(T)$ denote the family of all T -classes. Since T is reflexive, $x \in T(x)$ for every $x \in X$. In particular, $\mathcal{K}(T)$ is also a covering of X . Notice moreover

that, since T is symmetric, $x \in T(y)$ if and only if $y \in T(x)$ for every $x, y \in X$.

The following two easy lemmas, connecting classes and blocks, will be used later.

Lemma 3. *Let T be a tolerance on a set X . For every $B \subseteq X$:*

(i) *B is a T -preblock if and only if $B \subseteq \{x \in X \mid B \subseteq T(x)\}$.*

(ii) *B is a T -block if and only if $B = \{x \in X \mid B \subseteq T(x)\}$.*

Proof. A subset B of X is a T -preblock if and only if, for every $x \in B$, if $y \in B$, then $y \in T(x)$, i.e., if and only if $B \subseteq T(x)$ for every $x \in B$. This shows that B is a T -preblock if and only if

$$B \subseteq \{x \in X \mid B \subseteq T(x)\}.$$

This proves (i). Now, if B is a T -block and $y \in X$ is such that $B \subseteq T(y)$, then $B \cup \{y\}$ is a T -preblock containing B , and therefore $B = B \cup \{y\}$, i.e., $y \in B$. This shows that if B is a T -block, then the inclusion displayed above is an equality.

Conversely, if $B = \{x \in X \mid B \subseteq T(x)\}$, then B is a T -preblock by (i). Now, if B' is any T -preblock containing B , then $B \subseteq B' \subseteq T(x')$ for every $x' \in B'$, again by (i), and therefore $B' \subseteq \{x \in X \mid B \subseteq T(x)\} = B$. This shows that B is a T -block. \square

Lemma 4. *Let T be a tolerance on a set X .*

(i) *For every $x \in X$, $T(x)$ is the union of all T -blocks containing x .*

(ii) *Every T -block is the intersection of the T -classes of its elements.*

Proof. (i) By Lemma 1, $y \in T(x)$ if and only if there exists some T -block B such that $x, y \in B$.

(ii) Let B be a T -block. Then, $B \subseteq \bigcap_{x \in B} T(x)$, because $B \subseteq T(x)$ for every $x \in B$ by Lemma 3.(i). And if $y \in \bigcap_{x \in B} T(x)$, then $x \in T(y)$ for every $x \in B$, i.e., $B \subseteq T(y)$ and hence, by Lemma 3.(ii), $y \in B$. \square

As it was the case with blocks, not every covering of a set X is of the form $\mathcal{K}(T)$ for some tolerance T , as the following example shows.

Example 5. Let $X = \{1, 2, 3, 4, 5\}$ and $X_i = X - \{i\}$ for every $i = 1, \dots, 5$. Assume that there exists some tolerance T on X such that

$$\mathcal{K}(T) = \{X_1, \dots, X_5\}.$$

Then, $X_5 = T(i_0)$ for some $i_0 \in X_5$. It must happen therefore that $T(5) = X_{i_0}$ (because $(i_0, 5) \notin T$). Let $i_1 \neq i_0, 5$ and $X_{i_1} = T(i_2)$ for some $i_2 \in X - \{i_0, i_1, 5\}$. Then it also must happen that $X_{i_2} = T(i_1)$, again because $(i_1, i_2) \notin T$. Let finally i_3 be the element in X different from i_0, i_1, i_2 , and 5. Then, by exclusion, $T(i_3)$ must be X_{i_3} , but it is impossible, since $i_3 \notin X_{i_3}$.

This argument can be easily generalized to show that, for every *odd* integer $m \geq 3$, if $X = \{1, 2, \dots, m\}$ and $X_i = X - \{i\}$ for every $i = 1, \dots, m$, then there exists no tolerance T on X such that $\mathcal{K}(T) = \{X_1, \dots, X_m\}$. \square

But, contrary to what happens with blocks, a covering can be the family of classes of many different tolerances, as the following example shows. Here and henceforth we use undirected graphs to graphically represent tolerances in the way explained in the introduction.

Example 6. Let $X = \{1, 2, 3, 4\}$ and $X_i = X - \{i\}$ for every $i = 1, \dots, 4$. There are three possibilities for a tolerance T such that $\mathcal{K}(T) = \{X_1, X_2, X_3, X_4\}$:

- If $X_4 = T(1)$, then $X_1 = T(4)$ (because $(1, 4) \notin T$) and then $X_3 = T(2)$ and $X_2 = T(3)$ (because $T(i) \neq X_i$).
- If $X_4 = T(2)$, then $X_2 = T(4)$ and then $X_1 = T(3)$ and $X_3 = T(1)$.
- If $X_4 = T(3)$, then $X_3 = T(4)$ and then $X_1 = T(2)$ and $X_2 = T(1)$.

This shows that T can be any of the tolerances described in Figure 1.

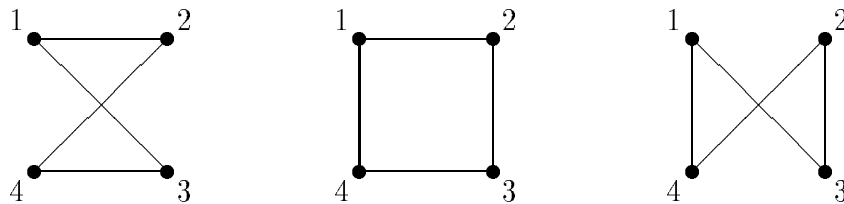


Fig. 1. Three tolerances with the same family of classes.

A similar argument proves that, for every *even* integer number $m \geq 2$, if $X = \{1, 2, \dots, m\}$ and $X_i = X - \{i\}$ for every $i = 1, \dots, m$, then there exist $(m-1)(m-3) \cdots 3$ tolerances T on X such that $\mathcal{K}(T) = \{X_1, \dots, X_m\}$. \square

It is interesting to notice that tolerances having the same families of classes have the same squares and they generate the same equivalence relations.

Lemma 7. *If T is a tolerance on a set X , then*

$$T^2 = \bigcup_{K \in \mathcal{K}(T)} K \times K.$$

Proof. A pair (x, y) belongs to T^2 if and only if there exists some $z \in X$ such that $(x, z), (z, y) \in T$, i.e., if and only if there exists some $z \in X$ such that $x, y \in T(z)$. \square

In the next result, given a tolerance T , we denote by T^* the equivalence relation it generates, i.e., its transitive closure.

Corollary 8. *If T_0 and T_1 are two tolerances on a set X such that $\mathcal{K}(T_0) = \mathcal{K}(T_1)$, then $T_0^2 = T_1^2$ and $T_0^* = T_1^*$.*

Proof. Last lemma entails that if $\mathcal{K}(T_0) = \mathcal{K}(T_1)$, then $T_0^2 = T_1^2$ and, being T_0 and T_1 reflexive, $T_0^* = (T_0^2)^* = (T_1^2)^* = T_1^*$, as we claimed. \square

The converse implication is false: both tolerances described in Figure 2 have the same squares, but different families of classes.

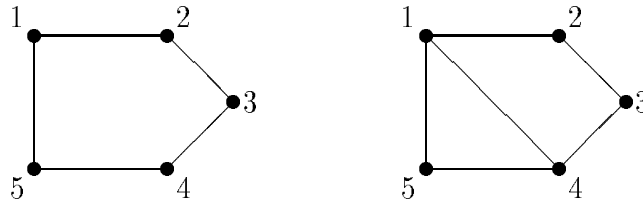


Fig. 2. $T_0^2 = T_1^2$ does not imply $\mathcal{K}(T_0) = \mathcal{K}(T_1)$.

3 Injective coverings and tolerances

In the next section we discuss the questions of whether or not a covering of a finite set X is of the form $\mathcal{K}(T)$ for some tolerance T , and whether or not it is of the form $\mathcal{K}(T)$ for more than one tolerance T . In this section, we show that to answer these questions (for arbitrary sets) we can restrict ourselves to what we shall call *injective coverings*.

Let \mathcal{C} be a covering of X . For every x of X , let

$$\mathcal{C}_x = \{C \in \mathcal{C} \mid x \in C\}.$$

The *equivalence associated* to \mathcal{C} is the relation

$$\theta(\mathcal{C}) = \{(x, x') \in X \times X \mid \mathcal{C}_x = \mathcal{C}_{x'}\}.$$

A covering \mathcal{C} of X is said to be *injective* when $\theta(\mathcal{C}) = \Delta_X$, the identity relation on X .

The following result is straightforward.

Proposition 9. *Let \mathcal{C} be a covering of X , θ the equivalence associated to it, and $\pi : X \rightarrow X/\theta$ the corresponding quotient mapping. Then:*

(i) *Every $C \in \mathcal{C}$ is a union of θ -classes, and in particular $\pi^{-1}\pi(C) = C$.*

(ii) *The family $\pi(\mathcal{C}) = \{\pi(C) \mid C \in \mathcal{C}\}$ is an injective covering of X/θ . \square*

On the other hand, the *equivalence associated* to a tolerance T on a set X is the relation

$$\theta(T) = \{(x, x') \in X \times X \mid T(x) = T(x')\}.$$

A tolerance T on X is said to be *injective* when $\theta(T) = \Delta_X$.

Lemma 10. *If T is an injective tolerance on a non-empty finite set X , then $\mathcal{K}(T)$ contains exactly $|X|$ elements.*

Proof. The mapping $X \rightarrow \mathcal{K}(T)$ defined by $x \mapsto T(x)$ is surjective by the definition of $\mathcal{K}(T)$ and injective by the injectivity of T . Therefore, it is bijective. \square

Given a tolerance on X , consider on the quotient set $X/\theta(T)$ the relation

$$\bar{T} = \{([x]_{\theta(T)}, [x']_{\theta(T)}) \mid (x, x') \in T\}.$$

It is well-defined, in the sense that if $[x_1]_{\theta(T)} = [x_2]_{\theta(T)}$ and $[x'_1]_{\theta(T)} = [x'_2]_{\theta(T)}$, then $(x_1, x'_1) \in T$ if and only if $(x_2, x'_2) \in T$: indeed, if $(x_1, x'_1) \in T$, then $x_1 \in T(x'_1) = T(x'_2)$, and hence $x'_2 \in T(x_1) = T(x_2)$, which implies that $(x_2, x'_2) \in T$. And, since T is a tolerance, it is clear that \bar{T} is also a tolerance, which we shall say to be *induced* by T .

Proposition 11. *Let T be a tolerance on a set X , let $\pi : X \rightarrow X/\theta(T)$ be the quotient mapping associated to $\theta(T)$, and let \bar{T} be the tolerance on $X/\theta(T)$ induced by T . Then:*

(i) $\theta(T) = \theta(\mathcal{K}(T)) = \theta(\mathcal{B}(T))$.

(ii) $\mathcal{K}(\bar{T}) = \pi(\mathcal{K}(T))$ and $\mathcal{B}(\bar{T}) = \pi(\mathcal{B}(T))$.

(iii) \bar{T} is injective.

Proof. (i) For every $x, x' \in X$,

$$\begin{aligned}
(x, x') \in \theta(T) &\iff T(x) = T(x') \\
&\iff \text{for every } y \in X, y \in T(x) \text{ if and only if } y \in T(x') \\
&\iff \text{for every } y \in X, x \in T(y) \text{ if and only if } x' \in T(y) \\
&\iff \text{for every } K \in \mathcal{K}(T), x \in K \text{ if and only if } x' \in K \\
&\iff \mathcal{K}(T)_x = \mathcal{K}(T)_{x'} \iff (x, x') \in \theta(\mathcal{K}(T)).
\end{aligned}$$

This shows that $\theta(T) = \theta(\mathcal{K}(T))$.

With respect to the equality $\theta(T) = \theta(\mathcal{B}(T))$, since $T(x) = \bigcup \mathcal{B}(T)_x$ for every $x \in X$ by Lemma 4.(i), it is clear that $\theta(\mathcal{B}(T)) \subseteq \theta(T)$. To prove the converse inclusion, let $(x, x') \in \theta(T)$. If B is any T -block containing x , then $B \subseteq T(x) = T(x')$ and hence, by Lemma 3.(ii), $x' \in B$. This shows that $\mathcal{B}(T)_x \subseteq \mathcal{B}(T)_{x'}$ and thus, by symmetry, $\mathcal{B}(T)_x = \mathcal{B}(T)_{x'}$, i.e., $(x, x') \in \theta(\mathcal{B}(T))$.

(ii) For every $x \in X$,

$$\begin{aligned}
\overline{T}([x]_{\theta(T)}) &= \{[y]_{\theta(T)} \mid ([x]_{\theta(T)}, [y]_{\theta(T)}) \in \overline{T}\} \\
&= \{[y]_{\theta(T)} \mid (x, y) \in T\} = \pi(T(x)).
\end{aligned}$$

This shows that $\mathcal{K}(\overline{T}) = \pi(\mathcal{K}(T))$.

As far as the equality $\mathcal{B}(\overline{T}) = \pi(\mathcal{B}(T))$ goes, notice on the one hand that if B is a T -block, then $\pi(B)$ is a \overline{T} -preblock. And $\pi(B)$ is actually a \overline{T} -block, because if \overline{B} is any \overline{T} -preblock containing it, then $\pi^{-1}(\overline{B})$ is a T -preblock containing B , and therefore $B = \pi^{-1}(\overline{B})$, which implies that $\pi(B) = \pi(\pi^{-1}(\overline{B})) = \overline{B}$. This proves the inclusion $\pi(\mathcal{B}(T)) \subseteq \mathcal{B}(\overline{T})$.

Conversely, if \overline{B} is any \overline{T} -block, then $\pi^{-1}(\overline{B})$ is a T -preblock, and is actually a T -block. Indeed, if B is a T -preblock containing $\pi^{-1}(\overline{B})$, then $\pi(B)$ is a \overline{T} -preblock containing \overline{B} , and thus $\pi(B) = \overline{B}$, from where we obtain by Proposition 9.(i), using that $\theta(T) = \theta(\mathcal{B}(T))$, that $B = \pi^{-1}(\pi(B)) = \pi^{-1}(\overline{B})$. Finally, since $\pi(\pi^{-1}(\overline{B})) = \overline{B}$, we have that $\overline{B} \in \pi(\mathcal{B}(T))$. This proves that $\mathcal{B}(\overline{T}) \subseteq \pi(\mathcal{B}(T))$.

(iii) By point (i) we have that $\theta(T) = \theta(\mathcal{K}(T))$ and $\theta(\overline{T}) = \theta(\mathcal{K}(\overline{T}))$, by point (ii) that $\theta(\mathcal{K}(\overline{T})) = \theta(\pi(\mathcal{K}(T)))$, and by Proposition 9.(ii) that $\theta(\pi(\mathcal{K}(T))) = \Delta_{X/\theta(\mathcal{K}(T))}$. \square

Corollary 12. *Let T be a tolerance on a set X . The following conditions are equivalent:*

(a) T is injective.

(b) The covering $\mathcal{K}(T)$ is injective.

(c) The covering $\mathcal{B}(T)$ is injective.

Proof. It is a direct consequence of Proposition 11.(i). \square

Corollary 13. *Let \mathcal{K} be a covering of a set X , let θ be the equivalence associated to \mathcal{K} and let $\pi : X \rightarrow X/\theta$ be the corresponding quotient mapping. Then, $\mathcal{K} = \mathcal{K}(T)$ for some tolerance T on X if and only if $\pi(\mathcal{K}) = \mathcal{K}(\hat{T})$ for some tolerance \hat{T} on X/θ .*

Proof. By Proposition 11, if $\mathcal{K} = \mathcal{K}(T)$, then $\pi(\mathcal{K}) = \mathcal{K}(\overline{T})$, where \overline{T} denotes the tolerance on $X/\theta = X/\theta(T)$ induced by T .

Conversely, assume that there exists a tolerance \hat{T} on X/θ such that $\pi(\mathcal{K}) = \mathcal{K}(\hat{T})$. Then

$$T = \{(x, y) \in X \times X \mid ([x]_\theta, [y]_\theta) \in \hat{T}\}$$

is a tolerance on X such that $T(x) = \pi^{-1}(\hat{T}([x]_\theta))$ for every $x \in X$. This implies that

$$\mathcal{K}(T) = \{\pi^{-1}(\widehat{K}) \mid \widehat{K} \in \mathcal{K}(\hat{T})\} = \{\pi^{-1}(\widehat{K}) \mid \widehat{K} \in \pi(\mathcal{K})\} = \mathcal{K},$$

where the last equality is a consequence of Proposition 9.(i). \square

Therefore, when discussing the main questions addressed in this paper and recalled at the beginning of this section, we can restrict ourselves to injective coverings:

- If we are able to decide whether an injective covering of a set is the family of classes of some tolerance or not, then we can decide the same question for an arbitrary covering \mathcal{C} of a set X . We only have to consider the covering $\pi(\mathcal{C})$ of $X/\theta(\mathcal{C})$ (which is injective): by Corollary 13, if it is the family of classes of some tolerance on $X/\theta(\mathcal{C})$, then \mathcal{C} is the family of classes of some tolerance on X , while if $\pi(\mathcal{C})$ is not the family of classes of some tolerance on $X/\theta(\mathcal{C})$, then neither is \mathcal{C} the family of classes of some tolerance on X .
- If we are able to decide whether an injective covering of a set is the family of classes of two different tolerances or not, then we can also decide the same question for an arbitrary covering \mathcal{C} of a set X . We only have to consider again the injective covering $\pi(\mathcal{C})$ of $X/\theta(\mathcal{C})$. If two different tolerances \hat{T}_0 and \hat{T}_1 on $X/\theta(\mathcal{C})$ such that $\pi(\mathcal{C}) = \mathcal{K}(\hat{T}_0) = \mathcal{K}(\hat{T}_1)$ exist, then (by Corollary

13, or rather its proof) \mathcal{C} is the family of classes of the tolerances

$$\begin{aligned} T_0 &= \{(x, y) \in X \times X \mid ([x]_{\theta(\mathcal{C})}, [y]_{\theta(\mathcal{C})}) \in \widehat{T}_0\} \\ T_1 &= \{(x, y) \in X \times X \mid ([x]_{\theta(\mathcal{C})}, [y]_{\theta(\mathcal{C})}) \in \widehat{T}_1\} \end{aligned}$$

which are clearly different. And if there exist two different tolerances T_0 and T_1 on X such that $\mathcal{C} = \mathcal{K}(T_0) = \mathcal{K}(T_1)$, then the tolerances \overline{T}_0 and \overline{T}_1 on $X/\theta(\mathcal{C})$ induced, respectively, by T_0 and T_1 are different and $\pi(\mathcal{C}) = \mathcal{K}(\overline{T}_0) = \mathcal{K}(\overline{T}_1)$ by Proposition 11.(ii).

Therefore, in the next section we shall only consider injective coverings.

4 When $\mathcal{K} = \mathcal{K}(T)$ for some T ?

We shall now answer the first question we proposed in the introduction: given a covering, when is it the family of classes of some tolerance T ?

In the sequel, let X be a *finite* set. For every covering \mathcal{C} of X , let $\mathcal{B}(\mathcal{C}) \subseteq \mathcal{P}(X)$ be the family of all subsets $B \subseteq X$ such that

$$B = \bigcap \{C \in \mathcal{C} \mid B \subseteq C\} \quad \text{and} \quad |\{C \in \mathcal{C} \mid B \subseteq C\}| = |B|.$$

The set $\mathcal{B}(\mathcal{C})$ can be constructed as follows. For each integer $n \geq 1$, we compute the intersections of all subfamilies of \mathcal{C} with n members. Then each of these intersections belongs to $\mathcal{B}(\mathcal{C})$ if and only if it has exactly n elements and it is contained only in those members of \mathcal{C} belonging to the subfamily whose intersection has produced it. This procedure can be simplified using the next lemma: once a member of $\mathcal{B}(\mathcal{C})$ is detected, no strict subset or superset of it can possibly belong to $\mathcal{B}(\mathcal{C})$.

Lemma 14. *Let \mathcal{C} be a covering of a finite set X . For every $B_1, B_2 \in \mathcal{B}(\mathcal{C})$, if $B_1 \subseteq B_2$, then $B_1 = B_2$.*

Proof. Assume that $B_1, B_2 \in \mathcal{B}(\mathcal{C})$ are such that $B_1 \subseteq B_2$, and thus

$$\{C \in \mathcal{C} \mid B_2 \subseteq C\} \subseteq \{C \in \mathcal{C} \mid B_1 \subseteq C\}.$$

Then $|B_1| \leq |B_2|$ and

$$|B_2| = |\{C \in \mathcal{C} \mid B_2 \subseteq C\}| \leq |\{C \in \mathcal{C} \mid B_1 \subseteq C\}| = |B_1|$$

and therefore $|B_1| = |B_2|$. Since X is finite, this implies that $B_1 = B_2$. \square

Example 15. Consider the covering

$$\mathcal{K} = \{X - \{i\} \mid i = 1, \dots, m\}$$

of the set $X = \{1, \dots, m\}$. Any intersection of a subfamily of k members of \mathcal{K} yields a $(m - k)$ -element subset of X :

$$\bigcap_{l=1}^k X - \{i_l\} = X - \{i_1, \dots, i_k\}.$$

If m is odd, then $k \neq m - k$ for any $k = 1, \dots, m$, and hence, in this case, $\mathcal{B}(\mathcal{K}) = \emptyset$. But, if m is even, then the intersection of any subfamily of $m/2$ members of \mathcal{K} has $m/2$ elements. On the other hand, every subset Y of X with $m/2$ elements is contained in exactly $m/2$ members of \mathcal{K} : namely, $X - \{i\}$ for every $i \notin Y$. This shows that when m is even, $\mathcal{B}(\mathcal{K})$ is the family of all subsets of X with exactly $m/2$ elements. \square

We introduce now two conditions that will play a relevant role in our main result.

Definition 16. Let \mathcal{B} and \mathcal{K} be two families of subsets of a set X .

(i) We shall say that \mathcal{B} satisfies the *block condition* (*BC*, for short) when, for each $Y \subseteq X$, if every pair of points in Y belong to some element of \mathcal{B} , then Y is contained in some element of \mathcal{B} .

(ii) We shall say that \mathcal{B} satisfies the *class condition* (*CC*, for short) for \mathcal{K} when, for each $K \in \mathcal{K}$, there exists some $x \in X$ such that $\bigcup \mathcal{B}_x = K$.

Notice that BC is condition (ii) in Theorem 2, which characterizes the families of blocks of tolerances.

Lemma 17. *Let X be a finite set, \mathcal{K} a covering of X with exactly $|X|$ members, and \mathcal{B} a family of subsets of X satisfying CC for \mathcal{K} . Then:*

(i) $\mathcal{K} = \{\bigcup \mathcal{B}_x \mid x \in X\}$.

(ii) For every $x, y \in X$, $\bigcup \mathcal{B}_x = \bigcup \mathcal{B}_y$ if and only if $x = y$.

Proof. Since \mathcal{B} satisfies CC for \mathcal{K} , we have that

$$\mathcal{K} \subseteq \{\bigcup \mathcal{B}_x \mid x \in X\},$$

and it is obvious that the right-hand side family has at most $|X|$ members, which by assumption is the cardinality of \mathcal{K} . Therefore

$$\mathcal{K} = \{\bigcup \mathcal{B}_x \mid x \in X\},$$

which proves (i). Moreover, in particular $\{\cup \mathcal{B}_x \mid x \in X\}$ has exactly $|X|$ members. This implies that, for every $x, y \in X$, $\cup \mathcal{B}_x = \cup \mathcal{B}_y$ if and only if $x = y$, which proves (ii). \square

Theorem 18. *Let T be an injective tolerance on a finite set X . Then:*

(i) $\mathcal{B}(T) \subseteq \mathcal{B}(\mathcal{K}(T))$.

(ii) $\mathcal{B}(T)$ satisfies BC, and it is maximal among all subfamilies of $\mathcal{B}(\mathcal{K}(T))$ satisfying BC.

(iii) $\mathcal{B}(T)$ satisfies CC for $\mathcal{K}(T)$, and it is maximal among all subfamilies of $\mathcal{B}(\mathcal{K}(T))$ satisfying CC for $\mathcal{K}(T)$.

Proof. To simplify the notations, we shall denote $\mathcal{B}(T)$ and $\mathcal{K}(T)$ simply by \mathcal{B} and \mathcal{K} , respectively.

(i) Let B be a T -block. By Lemma 3, we know that

$$B = \{x \in X \mid B \subseteq T(x)\}.$$

Then, since T is an injective tolerance,

$$|\{T(x) \mid B \subseteq T(x)\}| = |\{x \in X \mid B \subseteq T(x)\}| = |B|.$$

On the other hand, by Lemma 4.(ii) and Lemma 3 again, we have that

$$B = \bigcap \{T(x) \mid x \in B\} = \bigcap \{T(x) \mid B \subseteq T(x)\}.$$

This proves that $B \in \mathcal{B}(\mathcal{K})$.

(ii) We know from Theorem 2 that \mathcal{B} satisfies BC, and from (i) that $\mathcal{B} \subseteq \mathcal{B}(\mathcal{K})$. It remains to prove the maximality assertion.

So, let \mathcal{B}' be a subfamily of $\mathcal{B}(\mathcal{K})$ satisfying BC and such that $\mathcal{B} \subseteq \mathcal{B}'$: we want to prove that $\mathcal{B}' \subseteq \mathcal{B}$. Since \mathcal{B}' satisfies conditions (i) and (ii) in Theorem 2 (recall that $\mathcal{B}(\mathcal{K})$, and hence any subfamily of it, satisfies (i) by Lemma 14), there exists some tolerance S on X such that $\mathcal{B}(S) = \mathcal{B}'$.

To prove that $\mathcal{B}' \subseteq \mathcal{B}$, take any $B' \in \mathcal{B}'$. Let $x \in X$ be such that $B' \subseteq T(x)$. Then

$$B' \subseteq \cup \mathcal{B}_x \subseteq \cup \mathcal{B}'_x = S(x),$$

because $T(x) = \cup \mathcal{B}_x$ and $S(x) = \cup \mathcal{B}'_x$ by Lemma 4.(i), and $\mathcal{B} \subseteq \mathcal{B}'$. By Lemma 3.(ii), this implies that $x \in B'$. Therefore,

$$\{x \in X \mid B' \subseteq T(x)\} \subseteq B'.$$

Now, since $\mathcal{B}' \subseteq \mathcal{B}(\mathcal{K})$ and T is injective, we have that

$$|B'| = |\{T(x) \mid B' \subseteq T(x)\}| = |\{x \in X \mid B' \subseteq T(x)\}|$$

and thus

$$\{x \in X \mid B' \subseteq T(x)\} = B'.$$

Therefore, by Lemma 3, $B' \in \mathcal{B}$. This proves that $\mathcal{B}' \subseteq \mathcal{B}$.

iii) The equality $T(x) = \bigcup \mathcal{B}_x$ for every $x \in X$ (Lemma 4.(i)) implies that \mathcal{B} satisfies CC for \mathcal{K} , and from (i) we know that $\mathcal{B} \subseteq \mathcal{B}(\mathcal{K})$. We must again prove the maximality assertion. So, let \mathcal{B}' be a subfamily of $\mathcal{B}(\mathcal{K})$ satisfying CC for \mathcal{K} and such that $\mathcal{B} \subseteq \mathcal{B}'$: we want to prove that $\mathcal{B}' \subseteq \mathcal{B}$.

Let us show first that

$$(1) \quad T(x) = \bigcup \mathcal{B}'_x \quad \text{for every } x \in X.$$

Let $x \in X$. Then $T(x) = \bigcup \mathcal{B}_x$, and hence $T(x) \subseteq \bigcup \mathcal{B}'_x$, because $\mathcal{B} \subseteq \mathcal{B}'$. Since \mathcal{B}' is a subfamily of $\mathcal{B}(\mathcal{K})$ satisfying CC, Lemma 17.(i) implies then that there exists some $x_1 \in X$ such that $\bigcup \mathcal{B}'_{x_1} = T(x_1)$ and, again, $T(x_1) \subseteq \bigcup \mathcal{B}'_{x_1}$. If we iterate this procedure (setting $x = x_0$), we obtain an infinite sequence x_0, x_1, x_2, \dots of elements of X such that

$$\dots \bigcup \mathcal{B}'_{x_{n-1}} = T(x_n) \subseteq \bigcup \mathcal{B}'_{x_n} = T(x_{n+1}) \subseteq \bigcup \mathcal{B}'_{x_{n+1}} \dots$$

Since X is finite, there exist two elements x_n, x_m with $n < m$ such that $x_n = x_m$; moreover, we can choose n so that no x_i with $i < n$ appears twice in this sequence. Hence, $T(x_n) = T(x_m)$ and, in particular, $T(x_n) = \bigcup \mathcal{B}'_{x_n}$, because the sequence displayed above is non-decreasing.

If $n \geq 1$, we have that

$$\bigcup \mathcal{B}'_{x_{n-1}} = T(x_n) = \bigcup \mathcal{B}'_{x_n},$$

and then, by Lemma 17.(ii), $x_{n-1} = x_n$. This contradicts the choice of the index n . Therefore, $n = 0$, which provides the desired equality (because $x = x_0$) $T(x) = \bigcup \mathcal{B}'_x$. This completes the proof of assertion (1).

Now take any set $B' \in \mathcal{B}'$. Since $B' \in \mathcal{B}'_x$ for every $x \in B'$, assertion (1) above implies that $B' \subseteq \bigcup \mathcal{B}'_x = T(x)$ for every $x \in B'$, i.e.,

$$B' \subseteq \{x \in X \mid B' \subseteq T(x)\}.$$

On the other hand, we have that

$$|\{x \in X \mid B' \subseteq T(x)\}| = |\{T(x) \mid B' \subseteq T(x)\}| = |B'|,$$

because T is injective and, by assumption, $B' \in \mathcal{B}(\mathcal{K})$. Therefore,

$$\{x \in X \mid B' \subseteq T(x)\} = B'.$$

Then, by Lemma 3, B' is a T -block, i.e., $B' \in \mathcal{B}$. This entails that $\mathcal{B}' \subseteq \mathcal{B}$. \square

The following example shows that $\mathcal{B}(T)$ need not be the greatest subfamily of $\mathcal{B}(\mathcal{K}(T))$ satisfying BC and CC for $\mathcal{K}(T)$; actually, such a greatest subfamily may not even exist. This will be related to the fact that $\mathcal{K}(T)$ can be the family of classes of several tolerances.

Example 19. Consider the tolerances described in Figure 3. Then $\mathcal{K}(S) = \mathcal{K}(T)$, but

$$\begin{aligned} \mathcal{B}(S) &= \left\{ \{1, 4\}, \{1, 5\}, \{1, 6\}, \{2, 4\}, \{2, 5\}, \{2, 6\}, \{3, 4\}, \{3, 5\}, \{3, 6\} \right\} \\ \mathcal{B}(T) &= \left\{ \{1, 2, 3\}, \{1, 4\}, \{2, 5\}, \{3, 6\}, \{4, 5, 6\} \right\}. \end{aligned}$$

Then, by last theorem, $\mathcal{B}(S)$ and $\mathcal{B}(T)$ are maximal subfamilies of $\mathcal{K}(T) = \mathcal{K}(S)$ satisfying BC and CC for $\mathcal{K}(T)$, and they are incomparable. \square

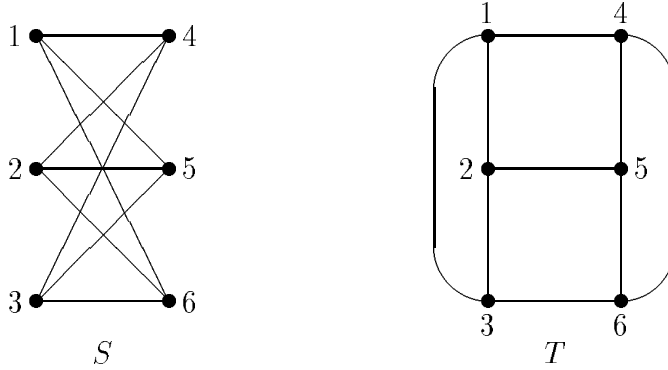


Fig. 3. The tolerances of Example 19.

The next proposition brings us closer to the desired characterization of those coverings that are families of tolerance classes.

Proposition 20. *Let X be a finite set and \mathcal{K} an injective covering of X with exactly $|X|$ members. Then the following conditions are equivalent:*

- (a) *There is an injective tolerance T on X such that $\mathcal{K}(T) = \mathcal{K}$.*
- (b) *There is an injective covering $\mathcal{B} \subseteq \mathcal{B}(\mathcal{K})$ satisfying BC and CC for \mathcal{K} .*
- (c) *There is a family $\mathcal{B} \subseteq \mathcal{B}(\mathcal{K})$ satisfying BC and CC for \mathcal{K} .*

Notice that, by Lemma 10 and Corollary 12, our assumption on the cardinality of \mathcal{K} is necessary if we want it to be the family of classes of some tolerance on

X .

Proof. (a) \implies (b). Let T be an injective tolerance on X . Then, by Corollary 12, $\mathcal{B}(T)$ is an injective covering of X and by Theorem 18, it is contained in $\mathcal{B}(\mathcal{K}(T))$ and it satisfies BC as well as CC for \mathcal{K} .

(b) \implies (c). Obvious.

(c) \implies (b). Any subfamily of $\mathcal{B}(\mathcal{K})$ satisfying CC for \mathcal{K} is an injective covering of X . Indeed, since \mathcal{K} is a covering of X , Lemma 17.(i) implies that every element x of X belongs to some member of \mathcal{B} . This shows that \mathcal{B} is a covering of X . And Lemma 17.(ii) implies that \mathcal{B} is injective: if $\mathcal{B}_x = \mathcal{B}_y$, then $\bigcup \mathcal{B}_x = \bigcup \mathcal{B}_y$ and hence $x = y$.

(b) \implies (a). Let $\mathcal{B} \subseteq \mathcal{B}(\mathcal{K})$ be an injective covering of X satisfying BC and CC for \mathcal{K} . The injectivity of \mathcal{B} and the satisfaction of BC imply, by Theorem 2 and Corollary 12, that there exists an injective tolerance T on X such that $\mathcal{B}(T) = \mathcal{B}$. Finally, Lemmas 4.(i) and 17.(i) imply that

$$\mathcal{K} = \{\bigcup \mathcal{B}_x \mid x \in X\} = \mathcal{K}(T).$$

□

We have shown that any subfamily \mathcal{C} of $\mathcal{B}(\mathcal{K}(T))$ that is a covering of X and that satisfies BC will be the family of blocks of a tolerance S . But the following example reveals that if \mathcal{C} does not satisfy CC for $\mathcal{K}(T)$, then $\mathcal{K}(S)$ need not be equal to $\mathcal{K}(T)$. This is true even if \mathcal{C} is a maximal subfamily of $\mathcal{B}(\mathcal{K}(T))$ satisfying BC.

Example 21. Consider the tolerance T described in Figure 4.

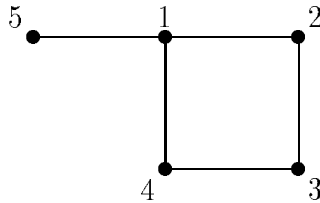


Fig. 4. The tolerance T of Example 21.

Then

$$\mathcal{K}(T) = \left\{ \{1, 2, 4, 5\}, \{1, 2, 3\}, \{2, 3, 4\}, \{1, 3, 4\}, \{1, 5\} \right\}$$

and

$$\mathcal{B}(\mathcal{K}(T)) = \left\{ \{1, 2\}, \{1, 3\}, \{1, 4\}, \{2, 3\}, \{2, 4\}, \{3, 4\}, \{1, 5\} \right\}.$$

The family

$$\mathcal{C} = \left\{ \{1, 3\}, \{1, 4\}, \{2, 3\}, \{2, 4\}, \{1, 5\} \right\}$$

is a maximal subfamily of $\mathcal{B}(\mathcal{K}(T))$ satisfying BC: it clearly satisfies BC, and if we add to it any other $\{a, b\}$ from $\mathcal{B}(\mathcal{K}(T))$, then the set $Y = \{c, a, b\}$, with c such that $\{c, a\}, \{c, b\} \in \mathcal{C}$, would violate BC. Actually, a similar argument easily shows that no subfamily of $\mathcal{B}(\mathcal{K}(T))$ with six or more elements satisfies BC. Thus, the largest number of elements of a subfamily of $\mathcal{B}(\mathcal{K}(T))$ satisfying BC is 5, the cardinality of \mathcal{C} .

The tolerance S such that $\mathcal{C} = \mathcal{B}(S)$ is described in Figure 5, and then

$$\mathcal{K}(S) = \left\{ \{1, 3, 4, 5\}, \{2, 3, 4\}, \{1, 2, 3\}, \{1, 2, 4\}, \{1, 5\} \right\} \neq \mathcal{K}(T) \quad \square.$$

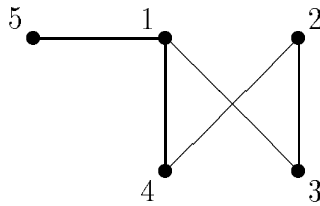


Fig. 5. The tolerance S of Example 21.

The existence of a subfamily of $\mathcal{B}(\mathcal{K})$ satisfying only BC does not imply that \mathcal{K} is the family of classes of some tolerance, as the following example shows.

Example 22. Let \mathcal{K} be the covering

$$\left\{ \{1, 2\}, \{1, 2, 3\}, \{2, 3, 4\}, \{1, 3, 4, 5\}, \{4, 5, 6\}, \{5, 6\} \right\}$$

of the set $X = \{1, 2, \dots, 6\}$. It is clearly injective, and a simple computation (using Lemma 14 to stop checking more intersections after having considered all intersections of pairs of members) shows that

$$\mathcal{B}(\mathcal{K}) = \left\{ \{1, 2\}, \{2, 3\}, \{1, 3\}, \{3, 4\}, \{4, 5\}, \{5, 6\} \right\}.$$

Then, for instance,

$$\mathcal{B} = \left\{ \{1, 2\}, \{2, 3\}, \{3, 4\}, \{4, 5\}, \{5, 6\} \right\}$$

is a (maximal) subfamily of $\mathcal{B}(\mathcal{K})$ satisfying BC.

Now, no subfamily of $\mathcal{B}(\mathcal{K})$ satisfying CC for \mathcal{K} exists, because $\{1, 3, 4, 5\} \in \mathcal{K}$ is not the union of any collection of members of $\mathcal{B}(\mathcal{K})$ containing a fixed element. In particular, there is no subfamily of $\mathcal{B}(\mathcal{K})$ satisfying BC and CC for \mathcal{K} . Thus, by Proposition 20, there does not exist any tolerance T on X such that $\mathcal{K} = \mathcal{K}(T)$. \square

Next theorem, our main result, shows that, contrary to what happens with BC, it is enough to have some subfamily in $\mathcal{B}(\mathcal{K})$ satisfying CC for \mathcal{K} to guarantee that \mathcal{K} is the family of classes of some tolerance.

Theorem 23. *Let X be a finite set and \mathcal{K} an injective covering of X with exactly $|X|$ members. Then the following conditions are equivalent:*

- (a) *There is a tolerance T on X such that $\mathcal{K}(T) = \mathcal{K}$.*
- (b) *There exists some family $\mathcal{B} \subseteq \mathcal{B}(\mathcal{K})$ satisfying CC for \mathcal{K} .*

Proof. The implication (a) \Rightarrow (b) is a consequence of Proposition 20. For the converse implication, assume that (b) holds, and consider the set of all subfamilies of $\mathcal{B}(\mathcal{K})$ satisfying CC for \mathcal{K} . This set is not empty, by (b), and finite, because $\mathcal{B}(\mathcal{K}) \subseteq \mathcal{P}(X)$ is finite. Therefore, there exists a maximal subfamily \mathcal{B} satisfying CC for \mathcal{K} .

Let us prove now that \mathcal{B} satisfies BC. To do so, consider any set $\overline{Y} \subseteq X$ such that every pair of elements of \overline{Y} belong to some member of \mathcal{B} . We want to prove that \overline{Y} is contained in some member of \mathcal{B} . Again, we can take a maximal subset $Y \subseteq X$ such that $\overline{Y} \subseteq Y$ and every pair of elements of Y belong to some member of \mathcal{B} . It is clearly enough to prove that Y is contained in some member of \mathcal{B} .

The assumption on Y clearly implies that

$$(2) \quad Y \subseteq \bigcup \mathcal{B}_y \quad \text{for every } y \in Y.$$

Let us show now that

$$(3) \quad \{\bigcup \mathcal{B}_y \mid y \in Y\} = \{K \in \mathcal{K} \mid Y \subseteq K\}.$$

The inclusion \subseteq is implied by Lemma 17.(i) and (2). To see the converse inclusion, take any set $K \in \mathcal{K}$ such that $Y \subseteq K$. By CC, $K = \bigcup \mathcal{B}_x$ for some $x \in X$. In particular, for every $y \in Y$, some $B \in \mathcal{B}$ exists such that $x, y \in B$. Then, $Y \cup \{x\}$ also satisfies that every pair of elements in it belong to some member of \mathcal{B} . By the maximality of Y , we obtain that $x \in Y$ and hence K belongs to the left-hand side set in equality (3). This proves this equality.

Let us prove now that $Y \in \mathcal{B}(\mathcal{K})$. By Lemma 17.(ii) and (3), we have the equality of cardinalities

$$|Y| = |\{K \in \mathcal{K} \mid Y \subseteq K\}|.$$

Considering the equality of sets

$$Y = \bigcap \{K \in \mathcal{K} \mid Y \subseteq K\},$$

notice that the inclusion \subseteq is obvious. To prove the converse inclusion, take any $x \in \bigcap \{K \in \mathcal{K} \mid Y \subseteq K\}$. Then, by (3), $x \in \bigcup \mathcal{B}_y$ for every $y \in Y$, and hence, for every $y \in Y$, there is some $B \in \mathcal{B}$ such that $x, y \subseteq B$. This implies that every pair of elements in $Y \cup \{x\}$ belong to some member of \mathcal{B} and then the maximality of Y implies again that $x \in Y$. This completes the proof that $Y \in \mathcal{B}(\mathcal{K})$.

Now, notice that the family $\mathcal{B}' = \mathcal{B} \cup \{Y\}$ satisfies CC for \mathcal{K} . More specifically, it turns out that $\bigcup \mathcal{B}'_x = \bigcup \mathcal{B}_x$ for every $x \in X$: if $x \notin Y$, it is obvious, and if $x \in Y$, then it follows from (2). Since \mathcal{B} is a maximal subfamily of $\mathcal{B}(\mathcal{K})$ satisfying CC for \mathcal{K} , this implies that $Y \in \mathcal{B}$. Summarizing, we have shown that every non-empty subset of X such that every pair of elements in it belongs to some member of \mathcal{B} , is contained in some member of \mathcal{B} . Thus \mathcal{B} satisfies BC.

We have thus obtained a subfamily $\mathcal{B} \subseteq \mathcal{B}(\mathcal{K})$ satisfying BC and CC for \mathcal{K} : by Proposition 20, this implies that $\mathcal{K} = \mathcal{K}(T)$ for some injective tolerance T on X . \square

Thus, an injective covering \mathcal{K} of a finite set X with the “right” number of members is the family of classes of some tolerance if and only if the family $\mathcal{B}(\mathcal{K})$ associated to it contains some subfamily satisfying CC for \mathcal{K} . A tolerance T on X such that $\mathcal{K} = \mathcal{K}(T)$ is then obtained as follows: any maximal subfamily \mathcal{B}_0 of $\mathcal{B}(\mathcal{K})$ satisfying CC for \mathcal{K} will be the family of blocks of a tolerance whose family of classes is exactly \mathcal{K} . Since the family of blocks characterizes a tolerance, we have the following corollary of this procedure.

Corollary 24. *Let X be a finite set and \mathcal{K} an injective covering of X with exactly $|X|$ members. There is a bijective correspondence between the set of tolerances T on X such that $\mathcal{K}(T) = \mathcal{K}$ and the set of maximal subfamilies of $\mathcal{B}(\mathcal{K})$ satisfying CC for \mathcal{K} .*

In particular, there exists one, and only one, tolerance T on X such that $\mathcal{K}(T) = \mathcal{K}$ if and only if there exists the greatest subset of $\mathcal{B}(\mathcal{K})$ satisfying CC for \mathcal{K} . \square

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