The conflict culminated in yet another public debate, this one between Tartaglia and Ferrari in Milan on August 10, 1548. Tartaglia later made much of Cardano's absence, blaming him for a cowardly decision "to avoid being present at the dispute." However, the contest, held on Ferrari's home turf, proved a failure for the visitor. Tartaglia blamed this on the rowdiness and partisanship of the crowd, whereas Ferrari naturally attributed the outcome to his own intellectual superiority. In any case, Tartaglia withdrew to return home, and Ferrari was proclaimed the brilliant victor. Mathematics historian Howard Eves, noting the hostile crowd and Ferrari's hot-headed reputation, says that Tartaglia may have been fortunate to escape alive.

These, then, were the events surrounding the solution of the cubic, a story at once complex, lusty, and absurd. It now remains for us to consider the great theorem at the heart of this strange tale.

Great Theorem: The Solution of the Cubic

Upon examining Chapter XI of *Ars Magna*, the modern reader has two surprises in store. One is that Cardano gave not a general proof but a specific example of a depressed cubic, namely

$$x^3 + 6x = 20$$

although in our discussion below we shall treat the more general

$$x^3 + mx = n$$

The second is that his argument was purely geometrical, involving literal cubes and their volumes. Actually, the surprise here is minimized when we recall the primitive state of algebraic symbolism and the exalted position of Greek geometry among Renaissance mathematicians.

The key result of Chapter XI is stated here in Cardano's own words, and his clever dissection of the cube is presented. His wordy "rule" for solving cubics at first sounds quite confusing, but recasting it in a more familiar, algebraic light shows that it does the job.

THEOREM Rule to solve $x^3 + mx = n$:

Cube one-third the coefficient of x; add to it the square of one-half the constant of the equation; and take the square root of the whole. You will duplicate [repeat] this, and to one of the two you add one-half the number you have already squared and from the other you subtract one-half the same . . . Then, subtracting the cube root of the first from the cube root of the second, the remainder which is left is the value of x.

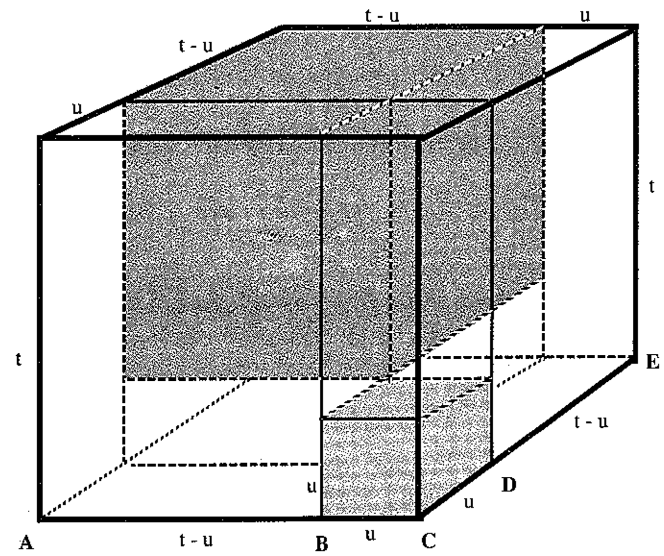


FIGURE 6.1

PROOF Cardano imagined a large cube, having side AC, whose length we shall denote by t, as shown in Figure 6.1. Side AC is divided at B into segment BC of length u and segment AB of length $t - u$. Here t and u are serving as auxiliary variables whose values we must find. As the diagram suggests, the large cube can be sliced into six pieces, each of whose volumes we now determine:

- a small cube in the lower front corner, with volume u^3
- a larger cube in the upper back corner, with volume $(t - u)^3$
- two upright slabs, one facing front along AB and the other facing to the right along DE, each with dimensions $t - u$ by u by t (the length of the side of the big cube) and thus each with volume $tu(t - u)$
- a tall block in the upper front corner, standing upon the small cube, with volume $u^2(t - u)$
- a flat block in the lower back corner, beneath the larger cube, with volume $u(t - u)^2$

Clearly the large cube's volume, t^3, equals the sum of these six component volumes. That is,

$$t^3 = u^3 + (t - u)^3 + 2tu(t - u) + u^2(t - u) + u(t - u)^2$$

Some rearrangement of these terms yields

$$(t-u)^3 + [2tu(t-u) + u^2(t-u) + u(t-u)^2] = t^3 - u^3$$

and factoring the common $(t-u)$ from the bracketed expression gives

$$(t-u)^3 + (t-u)[2tu + u^2 + u(t-u)] = t^3 - u^3 \qquad \text{or simply}$$

$$(t-u)^3 + 3tu(t-u) = t^3 - u^3 \qquad (*)$$

(The modern reader will notice that this equation can be derived instantly by simple algebra, without recourse to the arcane geometry of cubes and slabs. But this was not a route available to mathematicians in 1545.)

In (*) we have arrived at an equation reminiscent of the original cubic $x^3 + mx = n$. That is, if we let $t - u = x$, then (*) becomes $x^3 + 3tux = t^3 - u^3$, and this instantly suggests that we set

$$3tu = m \qquad \text{and} \qquad t^3 - u^3 = n$$

If we now can determine the quantities t and u in terms of m and n from the original cubic, then $x = t - u$ will yield the solution we seek.

Ars Magna does not present a derivation of these quantities. Rather, Cardano simply provided the specific rule for solving the "Cube and Cosa Equal to the Number" that was cited above. Trying to decipher his purely verbal recipe is no easy feat and certainly makes one appreciate the concise, direct approach of a modern algebraic formula. Exactly what was Cardano saying in this passage?

To begin, consider his two conditions on t and u, namely

$$3tu = m \qquad \text{and} \qquad t^3 - u^3 = n$$

From the former, we see that $u = m/3t$, and substituting this into the latter yields

$$t^3 - \frac{m^3}{27t^3} = n$$

Multiply both sides by t^3 and rearrange terms to get the equation

$$t^6 - nt^3 - \frac{m^3}{27} = 0$$

At first, this appears to be no improvement whatever, for we have traded our original third-degree equation in x for a sixth-degree equation in t. What saved the day, of course, was that the latter can be regarded as a *quadratic* equation in the variable t^3:

$$(t^3)^2 - n(t^3) - \frac{m^3}{27} = 0$$

The quadratic formula, which had been available to mathematicians for centuries and which we mentioned in the Epilogue to the previous chapter, then yielded:

$$t^3 = \frac{n \pm \sqrt{n^2 + \frac{4m^3}{27}}}{2}$$

$$= \frac{n}{2} \pm \frac{1}{2}\sqrt{n^2 + \frac{4m^3}{27}} = \frac{n}{2} \pm \sqrt{\frac{n^2}{4} + \frac{m^3}{27}}$$

Then, using only the positive square root, we have

$$t = \sqrt[3]{\frac{n}{2} + \sqrt{\frac{n^2}{4} + \frac{m^3}{27}}}$$

Now, we also know that $u^3 = t^3 - n$, and so we conclude that

$$u^3 = \frac{n}{2} + \sqrt{\frac{n^2}{4} + \frac{m^3}{27}} - n \qquad \text{or}$$

$$u = \sqrt[3]{-\frac{n}{2} + \sqrt{\frac{n^2}{4} + \frac{m^3}{27}}}$$

At last, we have the algebraic version of Cardano's rule for solving the depressed cubic $x^3 + mx = n$, namely

$$x = t - u$$

$$= \sqrt[3]{\frac{n}{2} + \sqrt{\frac{n^2}{4} + \frac{m^3}{27}}} - \sqrt[3]{-\frac{n}{2} + \sqrt{\frac{n^2}{4} + \frac{m^3}{27}}}$$

Q.E.D.

This expression is called a "solution by radicals" or an "algebraic solution" for the depressed cubic. That is, it involves only the original coefficients in the equation—that is, m and n—and the algebraic operations of addition, subtraction, multiplication, division, and extraction of roots, used only finitely often. A little study shows that this formula yields precisely the same result as Cardano's verbal "Rule" stated above.

Note that the key insight in Cardano's argument was to replace the solution of the cubic by the solution of a related quadratic equation (in t^3). He thus found a way to lower the problem by "one degree" and to move from the unfamiliar turf of cubics to the well-known realm of quadratics. This very clever process suggested a path to follow in attacking equations of the fourth, fifth, and higher degrees well.

As a concrete example, Cardano solved his prototype cubic $x^3 + 6x = 20$. According to his recipe, he first cubed a third of the coefficient of x to get $(\frac{1}{3} \times 6)^3 = 8$; next he squared half of the constant term (that is, half of 20) to get 100, and then added the 8, yielding a sum of 108 whose square root he took. To this he both added and subtracted half of the constant term, to get $10 + \sqrt{108}$ and $-10 + \sqrt{108}$, and finally his solution was the difference of cube roots of these two numbers:

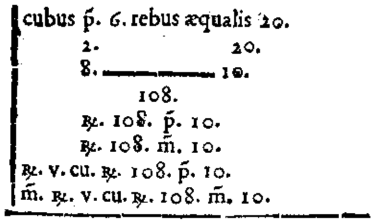
$$x = \sqrt[3]{10 + \sqrt{108}} - \sqrt[3]{-10 + \sqrt{108}}$$

Of course, we could simply substitute $m = 6$ and $n = 20$ into the pertinent algebraic formula. This yields

$$\sqrt{\frac{n^2}{4} + \frac{m^3}{27}} = \sqrt{108} \qquad \text{and so}$$

$$x = \sqrt[3]{10 + \sqrt{108}} - \sqrt[3]{-10 + \sqrt{108}}$$

which is clearly a "solution by radicals." It may come as a surprise—easily checked by a hand calculator—that this sophisticated-looking expression is nothing more than the number "2" in disguise, as Cardano correctly pointed out. One readily sees that $x = 2$ is indeed a solution of $x^3 + 6x = 20$.


cubus p̃. 6. rebus æqualis 20.
2. 20.
8. ——— 10.
108.
℞. 108. p̃. 10.
℞. 108. m̃. 10.
℞. v. cu. ℞. 108. p̃. 10.
m̃. ℞. v. cu. ℞. 108. m̃. 10.

Cardano's Rule for the cubic, from *Ars Magna* (photograph courtesy of Johnson Reprint Corporation)

Further Topics on Solving Equations

Observe that, having found one solution to the cubic, we are now in a position to find any others. For instance, since $x = 2$ solves the specific equation above, we know that $x - 2$ is one factor of $x^3 + 6x - 20$, and long division will generate the other, second-degree factor. In this case, $x^3 + 6x - 20 = (x - 2)(x^2 + 2x + 10)$. The solutions to the original cubic thus arise from solving the linear and quadratic equations

$$x - 2 = 0 \qquad \text{and} \qquad x^2 + 2x + 10 = 0$$

which is easily done. (This particular quadratic has no real solutions, so the cubic has as its only real solution $x = 2$.)

To the modern reader, the next two chapters of *Ars Magna* seem superfluous. Cardano titled Chapter XII "On the Cube Equal to the First Power and Number"—that is, $x^3 = mx + n$—and Chapter XIII was "On the Cube and Number Equal to the First Power"—that is, $x^3 + n = mx$. Today, we would regard these as having already been adequately covered by the formula above, for we would allow m and n to be negative. Mathematicians in the sixteenth century, however, demanded that all coefficients in the equation be positive. In other words, they regarded $x^3 + 6x = 20$ and $x^3 + 20 = 6x$ not just as different equations, but as intrinsically different *kinds* of equations. Such squeamishness about negative numbers is hardly surprising, given Cardano's tendency to think in terms of three-dimensional cubes, where sides of negative length make no sense. Of course, avoiding negatives led to a proliferation of cases and made *Ars Magna* considerably longer than we now find necessary.

So, Cardano could solve the depressed cubic in any of its three versions. But what about the *general* third-degree equation of the form $ax^3 + bx^2 + cx + d = 0$? It was Cardano's great discovery that, by means of a suitable substitution, this equation could be replaced by a related, depressed cubic that was, of course, susceptible to his formula. Before examining this "depressing" process for the cubic, we might take a quick look at it in a more familiar setting—as applied to solving quadratic equations:

Suppose we begin with the general second-degree equation

$$ax^2 + bx + c = 0 \qquad \text{where } a \neq 0$$

To depress it—that is, to eliminate its first-power term—we introduce the new variable y by substituting $x = y - b/2a$ to get

$$a\left(y - \frac{b}{2a}\right)^2 + b\left(y - \frac{b}{2a}\right) + c = 0 \qquad \text{which gives}$$

$$a\left(y^2 - \frac{b}{a}y + \frac{b^2}{4a^2}\right) + by - \frac{b^2}{2a} + c = 0 \qquad \text{or}$$

$$ay^2 - by + \frac{b^2}{4a} + by - \frac{b^2}{2a} + c = 0$$

Then, canceling the by terms, we get the depressed quadratic

$$ay^2 = \frac{b^2}{2a} - \frac{b^2}{4a} - c = \frac{2b^2}{4a} - \frac{b^2}{4a} - \frac{4ac}{4a} = \frac{b^2 - 4ac}{4a}$$

Hence

$$y^2 = \frac{b^2 - 4ac}{4a^2} \qquad \text{and} \qquad y = \frac{\pm\sqrt{b^2 - 4ac}}{2a}$$

Finally

$$x = y - \frac{b}{2a} = \frac{\pm\sqrt{b^2 - 4ac}}{2a} - \frac{b}{2a} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

which is of course the quadratic formula once again.

As this example suggests, depressing polynomials can prove quite useful. With this in mind, we return to Cardano's attack on the general cubic. Here, the key substitution is $x = y - b/3a$, which yields

$$a\left(y - \frac{b}{3a}\right)^3 + b\left(y - \frac{b}{3a}\right)^2 + c\left(y - \frac{b}{3a}\right) + d = 0$$

Upon expanding, this becomes

$$\left(ay^3 - by^2 + \frac{b^2}{3a}y - \frac{b^3}{27a^2}\right) + \left(by^2 - \frac{2b^2}{3a}y + \frac{b^3}{9a^2}\right) + \left(cy - \frac{cb}{3a}\right) + d = 0$$

There is but one critical observation we need to make regarding this blizzard of letters, namely, that the y^2 terms will cancel out. Thus, the new cubic loses its second-degree term (as desired). If we divide through by a, the resulting equation takes the form $y^3 + py = q$. We solve this for y by Cardano's formula and from there have no difficulty in determining $x = y - b/3a$.

To see this process in action, consider the cubic

$$2x^3 - 30x^2 + 162x - 350 = 0$$

With the substitution $x = y - b/3a = y - (-30/6) = y + 5$, we get

$$2(y + 5)^3 - 30(y + 5)^2 + 162(y + 5) - 350 = 0$$

which becomes

$$2y^3 + 12y - 40 = 0 \qquad \text{or simply} \qquad y^3 + 6y = 20$$

But this is, of course, the very depressed cubic we solved earlier, and so we know that $y = 2$. Hence $x = y + 5 = 7$, and this checks in the original equation.

Ars Magna did not handle the general cubic quite so concisely as we did here. Instead, demanding only positive coefficients, Cardano had to wade through a string of different cases, such as "On the Cube, Square, and First Power Equal to the Number," "On the Cube Equal to the Square, First Power, and Number," "On the Cube and Number Equal to Square, First Power, and Number," "On the Cube Equal to the Square and First Power," and so on. At last, 13 chapters after solving the depressed cubic, he brought the matter to its conclusion. The cubic had been solved.

Or had it? Although Cardano's formula seemed to be an amazing triumph, it introduced a major mystery. Consider, for instance, the depressed cubic $x^3 - 15x = 4$.

Using $m = -15$ and $n = 4$ in the formula developed above, we get

$$x = \sqrt[3]{2 + \sqrt{-121}} - \sqrt[3]{-2 + \sqrt{-121}}$$

Obviously, if negative numbers were suspect in the 1500s, their *square roots* seemed absolutely preposterous, and it was easy to dismiss this as an unsolvable cubic. Yet it can easily be checked that the cubic above has three different and perfectly real solutions: $x = 4$ and $x = -2 \pm \sqrt{3}$. What was Cardano to make of such a situation—the so-called "irreducible case of the cubic"? He took a few half-hearted stabs at investigating what we now call "imaginary" or "complex" numbers but ultimately dismissed the whole enterprise as being "as subtle as it is useless."

It would be another generation before Rafael Bombelli (ca. 1526–1573), in his 1572 treatise *Algebra*, took the bold step of regarding imaginary numbers as a necessary vehicle that would transport the mathematician from the *real* cubic equation to its *real* solutions; that is, while we begin and end in the familiar domain of real numbers, we seem compelled to move into the unfamiliar world of imaginaries to complete our journey. To mathematicians of the day, this seemed incredibly strange.

We shall examine briefly what Bombelli did. Temporarily disregarding any latent prejudice against $\sqrt{-1}$, we cube the expression $2 + \sqrt{-1}$ to get

$$\begin{aligned}(2 + \sqrt{-1})^3 &= 8 + 12\sqrt{-1} - 6 - \sqrt{-1} \\ &= 2 + 11\sqrt{-1} = 2 + \sqrt{-121}\end{aligned}$$

But if $(2 + \sqrt{-1})^3 = 2 + \sqrt{-121}$, then it surely makes sense to say that

$$\sqrt[3]{2 + \sqrt{-121}} = 2 + \sqrt{-1}$$

Similarly, we can see that $\sqrt[3]{-2 + \sqrt{-121}} = -2 + \sqrt{-1}$. Then, reexamining the cubic $x^3 - 15x = 4$, Bombelli arrived at the solution

$$\begin{aligned}x &= \sqrt[3]{2 + \sqrt{-121}} - \sqrt[3]{-2 + \sqrt{-121}} \\ &= (2 + \sqrt{-1}) - (-2 + \sqrt{-1}) = 4\end{aligned}$$

which is correct!

Admittedly, Bombelli's technique raised more questions than it resolved. For one thing, how does one know beforehand that $2 + \sqrt{-1}$ is going to be the cube root of $2 + \sqrt{-121}$? It would not be until the middle of the eighteenth century that Leonhard Euler could give a sure-fire technique for finding roots of complex numbers. Furthermore, what exactly were these imaginary numbers, and did they behave like their real cousins?

It is true that the full importance of complex numbers did not become evident until the work of Euler, Gauss, and Cauchy more than two centuries later, and we shall meet this topic again in the Epilogue to Chapter 10. Still, Bombelli deserves credit for recognizing that such numbers have a role to play in algebra, and he thereby stands as the last in the line of the great Italian algebraists of the sixteenth century.

One point should be stressed here. Contrary to popular belief, imaginary numbers entered the realm of mathematics not as a tool for solving *quadratics* but as a tool for solving *cubics*. Indeed, mathematicians could easily dismiss $\sqrt{-121}$ when it appeared as a solution to $x^2 + 121 = 0$ (for this equation clearly has no real solutions). But they could not so easily ignore $\sqrt{-121}$ when it played such a pivotal role in yielding the solution $x = 4$ for the previous cubic. So it was cubics, not quadratics, that gave complex numbers their initial impetus and their now-undisputed legitimacy.

We should make a final observation about *Ars Magna*. In Chapter XXXIX, Cardano introduced the solution of the quartic with the words:

> There is another rule, more noble than the preceding. It is Lodovico Ferrari's, who gave it to me on my request. Through it we have all the solutions for equations of the fourth power.

While the procedure is quite complicated, its two key steps should ring a bell:

1. Beginning with a general quartic $ax^4 + bx^3 + cx^2 + dx + e = 0$, depress it using the substitution $x = y - b/4a$ and then divide through by a, to generate a depressed quartic in y:

$$y^4 + my^2 + ny = p$$

2. By cleverly introducing auxiliary variables, replace this quartic by a related *cubic*, which then can be solved using the techniques developed above. Here again, Ferrari invoked the rule-of-thumb that the way to solve an equation of a given degree is to reduce it to the solution of an equation of one degree less.

Those who were capable of reading through this, and all of the other discoveries in *Ars Magna*, must have been breathless by the time they finished. The art of equation solving had been taken to new heights, and Luca Pacioli's original assessment that cubics, let alone quartics, were beyond the reach of algebra had been shattered. It is little wonder that Cardano ended his book with the enthusiastic and rather touching statement: "Written in five years, may it last as many thousands."

Epilogue

One question that the Cardano-Ferrari work left unanswered was the algebraic solution of the quintic, or fifth-degree, equation. Their efforts certainly suggested that such a solution by radicals was possible and