1 Euler's proof of the infinity of primes

Let's begin by Euler's proof that there are an infinity of primes, a very 'eulerian' proof.

If p is prime, then $(1/p) < 1$ and the sum of the geometric progression is

$$\sum_{k=0}^{\infty} \frac{1}{p^k} = \frac{1}{1 - \frac{1}{p}}.$$

Let's assume that the primes form a finite set $\mathbf{P} = \{p_1, p_2, \ldots p_n\}$. Let's do the products of all the sums of the geometric progressions (Euler loved these kinds of algebraic malabarisms). It is not to difficult to see that the result is

$$\prod_{i=1}^{n} (\sum_{k=0}^{\infty} \frac{1}{p_i^k})$$

that is precisely the sum of all the fractions where the denominators are f the form $p_1^{x_1} p_2^{x_2} p_3^{x_3} \ldots p_n^{x_n}, x_i = 1, 2, \ldots \infty$ and every combination of x_i appears only once. This means that

$$\prod_{i=1}^{n} (\sum_{k=0}^{\infty} \frac{1}{p_i^k}) = \sum_{n=1}^{\infty} \frac{1}{n}$$

which is divergent, but

$$\prod_{i=1}^{n} (\sum_{k=0}^{\infty} \frac{1}{p_i^k}) = \prod_{i=1}^{n} \frac{1}{1 - \frac{1}{p_i}}$$

is clearly inite, and we reach a contradiction.

Sum of the inverse of the primes

Let N be a positive integer. Every $n \leq N$ is a unique product of primes p, $p \leq n \leq N$. Also, for each prime p,

$$\sum_{k=1}^{\infty} \frac{1}{p^k} = \frac{1}{1 - \frac{1}{p}}$$

Therefore, as seen above

$$\sum_{n=1}^{N} \frac{1}{n} \le \prod_{p\le N} (\sum_{k=1}^{\infty} \frac{1}{p^k}) = \prod_{p\le N} \frac{1}{1-\frac{1}{p}}$$

To eliminate the product we apply logarithms

$$\log \prod_{p\le N} \frac{1}{1-\frac{1}{p}} = -\sum_{p\le N} \log(1-\frac{1}{p}).$$

Expanding $\log(1-1/p)$ by the expansion of $\log(1+x)$ we have that for each prime p

$$-\log(1-\frac{1}{p}) = \sum_{m=1}^{\infty} \frac{1}{mp^m}$$

now

$$\begin{aligned} \sum_{m=1}^{\infty} \frac{1}{mp^m} &\le \frac{1}{p} + \frac{1}{2p^2} (\sum_{h=0}^{\infty} \frac{1}{p^h}) = \\ &= \frac{1}{p} + \frac{1}{2p^2} \frac{1}{1-\frac{1}{p}} = \frac{1}{p} + \frac{1}{2p(p-1)} < \frac{1}{p} + \frac{1}{p^2}. \end{aligned}$$

So

$$\log \sum_{n=1}^{N} \frac{1}{n} \le \log \prod_{p\le N} \frac{1}{1-\frac{1}{p}} \le \sum_{p\le N} \frac{1}{p} + \sum_{p\le N} \frac{1}{p^2} \le \sum_{p} \frac{1}{p} + \sum_{n=1}^{\infty} \frac{1}{n^2}$$

But, $\sum_{n=1}^{\infty}(1/n^2) = (\pi^2/6)$ —one of Euler's stellar results— and if we make N tend to infinity, $\sum(1/p)$ must be divergent.