P systems with vague boundaries: the t-norm approach

Stefano Aguzzoli*, Daniela Besozzi, Brunella Gerla***, Corrado Manara*****

*Università degli Studi di Milano
Dipartimento di Scienze dell'Informazione
Via Comelico 39, 20135 Milano, Italy
E-mail: `aguzzoli@dsi.unimi.it`

**Università degli Studi di Milano
Dipartimento di Informatica e Comunicazione
Via Comelico 39, 20135 Milano, Italy
E-mail: `besozzi@dico.unimi.it`

***Università degli Studi di Salerno
Dipartimento di Matematica e Informatica
Via Ponte don Melillo, 84084 Fisciano (SA), Italy
E-mail: `{bgerla,cmanara}@unisa.it`

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1 Introduction

In the everyday life we solve a lot of problems not caring at all about *precision* and perfection of the solution. The increase of precision leads to an increase of the amount of information whose *significance* then decreases until a point is reached, after which precision and significance are mutually excluding characteristics. Then, imprecision (vagueness) cannot be avoided and often is necessary to convey relevant information [4].

This vagueness, called *fuzziness*, can raise during the process of grouping objects having some property P. In general, P cannot characterize unambiguously the group of objects because there can exist some *borderline* elements, which make unsharp the boundaries of the set

$$X = \{x \mid x \text{ has the property } P\}.$$

This led to the development of *fuzzy set theory* and *fuzzy logic* (see [8, 9, 7] and references therein).

Since the introduction of the notion of fuzzy set , the term "fuzzy logic" has been largely used but it is important to make some distinctions. In its *wide sense*, fuzzy logic is a synonymous of fuzzy set theory. In its *narrow sense*, it can be considered as a logical system which aims at a formalization of approximate reasoning. Fuzzy logic originates with the attempt to handle concepts which admit many (more than two) degrees of truth and it is based on a *comparative notion of truth*: one statement may be more true than another one. From this point of view fuzzy logic is worth studying [2].

The path from initial considerations about fuzziness to a formal logical system is not straightforward. Nowadays, the various approaches to *many-valued logics* found in the literature are competing as natural candidates to offer to the engineering discipline of fuzzy logic the theoretical foundations that have been lacking for several years.

Fuzzy logic is naturally described as the logic of degrees of truth, thus differentiating itself from logics of belief, and from probabilistic logic and modal logic, which are not truth-functional. Connectives of a logic behave truth-functionally when the value of the connection of some propositions is a function of the value of the same propositions only.

In order to set a formal framework to deal with fuzzy or uncertain reasoning, if we require the set of truth-values to be linearly ordered, and the connectives of the logic to be truth-functional, then a major tool used in fuzzy logic for modelling uncertain information is the definition of suitable *triangular norms*, *t-norms* for short [3].

In this work we propose a t-norm based approach for handling imprecision in *P systems*. P systems, initially proposed in [5], are a class of distributed and parallel computing devices inspired by the architecture of living cells and the way biological substances are both modified and moved among internal organelles. In a P system, each compartment (an organelle inside the cell) can be seen as a computing unit, having its own data and its local program (molecular substances and biochemical reactions), and all compartments considered as a whole (the cell) can be seen as an "unconventional" computing device. In particular, each compartment is delimited and separated from the rest by a membrane; the whole computing unit is formally characterized by a membrane structure, where membranes can be hierarchically placed inside a unique external membrane delimiting the entire cell. All membranes are semi-permeable barriers, which either allow some substances to move inwards or outwards, and consequently change their location in the membrane structure, or block the movement of some other substances. The biological substances and reactions are represented by means of objects and evolution rules. Objects are usually symbols or strings over a given alphabet, evolution rules are given as rewriting rules with target indications, thus describing both the transformation and the communication of objects.

A computation in P systems is obtained by starting from an initial configuration, identified by the membrane structure, the objects and the rules initially present inside it, and then letting the system evolve. The application of rules is performed in a nondeterministic and maximal parallel manner: all the applicable rules have to be used to modify all objects which can be the subject of a rule, and this is done in parallel for all membranes (a universal clock is assumed to exist). Whenever no rule can be further applied, the computation halts and the output is defined in terms of the objects sent out the external membrane or, alternatively, collected inside a specified membrane. No output is obtained if the computation never halts (that is, whenever a rule can be continuously applied).

Further notions on many variants of P systems, as well as an updated bibliography, can be found in [6] and at `http://psystems.disco.unimib.it`.

2 Triangular norms

Definition 2.1 (t-norm) A t-norm is any operator $\overset{\wedge}{*}$: $[0,1]^2 \to [0,1]$ satisfying:

1. *Associativity*: $x \overset{\wedge}{*} (y \overset{\wedge}{*} z) = (x \overset{\wedge}{*} y) \overset{\wedge}{*} z$.
2. *Commutativity*: $x \overset{\wedge}{*} y = y \overset{\wedge}{*} x$.
3. *Monotonicity* in each argument: If $y_1 \leq y_2$ then $x \overset{\wedge}{*} y_1 \leq x \overset{\wedge}{*} y_2$. If $x_1 \leq x_2$ then $x_1 \overset{\wedge}{*} y \leq x_2 \overset{\wedge}{*} y$.
4. *Absorption*: $x \overset{\wedge}{*} 0 = 0$ and *Unity*: $x \overset{\wedge}{*} 1 = x$.

A t-norm $\overset{\wedge}{*}$ is *continuous* if it is continuous as a real-valued function with respect to each variable.

A *t-conorm* is the dual operator of a t-norm.

Definition 2.2 (t-conorm) A t-conorm is any operator $\overset{*}{\vee}$: $[0,1]^2 \to [0,1]$ satisfying:

1. *Associativity*: $x \overset{*}{\vee} (y \overset{*}{\vee} z) = (x \overset{*}{\vee} y) \overset{*}{\vee} z$.
2. *Commutativity*: $x \overset{*}{\vee} y = y \overset{*}{\vee} x$.
3. *Monotonicity* in each argument: If $y_1 \leq y_2$ then $x \overset{*}{\vee} y_1 \leq x \overset{*}{\vee} y_2$. If $x_1 \leq x_2$ then $x_1 \overset{*}{\vee} y \leq x_2 \overset{*}{\vee} y$.
4. *Absorption*: $x \overset{*}{\vee} 0 = x$ and *Unity*: $x \overset{*}{\vee} 1 = 1$.

As we can note, t-norms and t-conorms differ only in the boundary condition imposed.

Triangular norms can be used to model *graded-truth conjunction*. Some natural requirements such a conjunction should satisfy are met by the definition of t-norm. Indeed, the truth degree of the conjunction of propositions

A and B should not depend on the order in which A and B are connected. The same is true for the truth degree of conjunctions of several propositions $A_1, A_2, \ldots, A_u$. These two properties are witnessed by commutativity and associativity. It is also natural to assume that the truth degree of the conjunction of a proposition with a complete falsity should be completely false, thus justifying the absorption requirement. Analogously, the conjunction of a proposition with a complete truth should not have smaller truth degree than the proposition has. Finally, we should expect that high truth degrees of propositions A and B would correspond to a high truth degree of their conjunction, and this is assured by the fact that t-norms are increasing functions in each argument. Note also that the absorption and unity properties state that each t-norm coincides with the conjunctive connective of classical logic when properly restricted to the domain $\{0,1\}^2$.

There exist uncountably many t-norms. If we restrict our attention to continuous t-norms only, we shall see that there exist three main t-norms, all the others arising as suitable combinations of them:

- Łukasiewicz t-norm: $x \odot y = \max(0, x+y-1)$.
- Gödel t-norm: $x \wedge y = \min(x, y)$.
- Product t-norm: $x \cdot y = xy$, product of real numbers.

An analogous approach to *graded-truth implication* requires that the truth degree of A *implies* B should be high when the truth degree of A is not significantly higher than the truth degree of B: then any binary operator $\Rightarrow$, chosen as semantics of an implication connective, should be non-increasing in its first argument and non-decreasing in the second one. To model a sound and powerful rule of *graded-truth modus ponens*, we require that from lower bounds a, c of the truth degrees of propositions A and $A \Rightarrow B$ respectively, we can infer a lower bound b for the truth-degree of B. If we combine a and c by some fixed t-norm $\overset{\wedge}{*}$, then we may require $c = a \Rightarrow b$ to be the maximum value such that $a \overset{\wedge}{*} c \leq b$ is satisfied. Actually, the following lemma holds:

Lemma 2.1 *Given any continuous t-norm $\overset{\wedge}{*}$, there is a unique operator $\Rightarrow_{\overset{\wedge}{*}}: [0,1]^2 \to [0,1]$, such that, for all $x, y, z \in [0,1]$:*

$$x \overset{\wedge}{*} z \leq y \qquad \text{if and only if} \qquad z \leq x \Rightarrow_{\overset{\wedge}{*}} y.$$

The operator $\Rightarrow_{\overset{\wedge}{*}}$ is called the *residuum* of $\overset{\wedge}{*}$ and is defined by:

$$x \Rightarrow_{\overset{\wedge}{*}} y = \max(z | x \overset{\wedge}{*} z \leq y).$$

For any continuous t-norm, the residuum operation coincides with the truth-table of classical implication, when its domain is restricted to $\{0,1\}^2$.

The residuum operators induced by the three main continuous t-norms are:

- Łukasiewicz implication: $x \Rightarrow_{\odot} y = \min(1, 1 - x + y)$.
- Gödel implication: $x \Rightarrow_{\wedge} y = \begin{cases} 1 & \text{if } x \leq y \\ y & \text{otherwise.} \end{cases}$
- Product implication: $x \Rightarrow_{.} y = \begin{cases} 1 & \text{if } x \leq y \\ y/x & \text{otherwise.} \end{cases}$

The choice of a (continuous) t-norm $\overset{\wedge}{*}$ determines an entire propositional many-valued logic, with its connectives of conjunction, implication, negation, and modus ponens.

Hájek's Basic Logic BL [2], which is presented as a traditional Hilbert system with a finite set of axiom schemata, is the logic of all continuous t-norms and their residua. That is, BL proves a formula φ iff the standard interpretation of φ evaluates identically to 1, in each t-norm algebra

$$([0,1], \overset{\wedge}{*}, \Rightarrow_{\overset{\wedge}{*}}, 0).$$

The class of all algebraic models of BL forms the algebraic variety $\mathcal{BL}$. The study of subvarieties of $\mathcal{BL}$ is the main tool to derive results in all the most important many-valued logics, as Łukasiewicz, Gödel, and Product logics. These results concern both logical and complexity aspects: for the latter, the study of free algebras is of the foremost importance.

Łukasiewicz logic is unique among many-valued propositional logics because all its connectives (primitive and derived) have continuous functions as their semantics.

3 P systems with vague boundaries

We assume the reader is familiar with the basic notions and notations of P systems.

We briefly recall that a *membrane structure* consists of a set of membranes hierarchically embedded in a unique membrane, called the *skin membrane*. The membrane structure is identified with a string of correctly matching square parentheses, placed in a unique pair of matching parentheses; each pair of matching parentheses corresponds to a membrane. Each membrane identifies a region, delimited by it and the membranes (if any) immediately inside it. Usually, a unique label is univocally associated to each membrane. An *object* can be a symbol or a string over a specified finite alphabet V; *multisets* of objects are usually considered in order to describe the presence of multiple copies of any given object. In the following, we will only consider structured objects, that is strings. Objects are modified by means of *evolution rules* which are, usually, context-free rewriting rules with an associated target indication (*tar*, in short) of the form $here, out, in$. The target indication determines the region where the object is communicated after the application of the rule: if $tar = here$, then the object remains in the same region; if $tar = out$, then the object exits from the region where it was placed; if $tar = in$, then the object nondeterministically enters one of the membranes immediately inside the region where the rule is applied, if any inner region exists (otherwise the rule cannot be applied).

In this section we introduce the notion of a P system with *vague boundaries*, which satisfies some peculiar aspects not common with the classical definition of P system:

- each object can be simultaneously present inside many regions, this is formally expressed by assigning a membership value to it, denoting "how much it belongs" to every region;
- each rule can be simultaneously active in many regions, this is formally expressed by assigning a value to it, denoting "how much it is active" inside every region;
- there is no crisp separation of regions, instead each membrane represents a vague boundary with respect to the adjacent regions.

As a consequence, we believe that the communication of objects can be described with a t-norm approach (by evaluating the composition of the truth degree of the objects with the truth degree of the rules) and it is no more necessary to associate target indications to rules.

Formally, a P system with vague boundaries in the t-norm approach is defined as

$$\Pi = (V, T, \mu, M, R, \Phi, (\overset{\wedge}{*}, \overset{*}{\vee}), i_o)$$

where:

- V is the alphabet of the system;
- $T \subseteq V$ is the terminal (or output) alphabet;
- μ is a membrane structure consisting of n membranes, which are injectively labelled by numbers in the set $\{1, \ldots, n\}$;
- $M = \{\sigma_1, \ldots, \sigma_p\}$ is a (multi)set of strings over V, representing the objects initially present in all regions of the system;
- $R = \{r_1, \ldots, r_q\}$ is a finite set of context-free rewriting rules of the form $a \to x$, with $a \in V, x \in V^*$, associated with the regions of μ;
- $\Phi = (\mu_1, \ldots, \mu_n)$ is the membership function initially associated with the regions of μ, where $\mu_i : M \cup R \to [0, 1]$ for all $i = 1, \ldots, n$;

- $(\overset{\wedge}{*}, \overset{*}{\vee})$ is the chosen pair of t-norm and t-conorm;
- i_o is a number in the set $\{1, \ldots, n\} \cup \{\infty\}$, indicating the *output* region.

We denote by m_i the membrane (and its corresponding region) labelled with number i, $i = 1, \ldots, n$, present in the membrane structure μ. Note that, since we do not consider any dissolving or dividing action for membranes, the membrane structure will never be modified during any computation.

As in classical rewriting P systems, for each string that can be the subject of many evolution rules at the same time (possibly applicable on more than one symbol in the string), we consider only one possibility to rewrite it: we apply only one evolution rule (nondeterministically chosen among all applicable rules) and we apply it over only one symbol in the string (nondeterministically chosen among all rewritable symbols). Hence, no parallel rewriting methods will be used here.

We consider the proposition "The string σ_j is in the region delimited by m_i" for every m_i in μ, $\sigma_j \in M$, $j = 1, \ldots, p$, and we denote it by $\mu_i(\sigma_j)$. In the same way, we denote by $\mu_i(r_k)$ the proposition "The rule r_k is active in the region delimited by m_i" for every m_i in μ, $r_k \in R$, $k = 1, \ldots, q$. Hence, we have:

$$\mu_i(\sigma_j) \in [0,1], \quad \forall i \in \{1, \ldots, n\} \; \forall \sigma_j \in M,$$

$$\mu_i(r_k) \in [0,1], \quad \forall i \in \{1, \ldots, n\} \; \forall r_k \in R.$$

Consider two configurations $C^{(t)} = (\mu, M^{(t)})$ of Π at time t and $C^{(t+1)} = (\mu, M^{(t+1)})$ of Π at time $t+1$. For every $\sigma \in M^{(t+1)}$ let

$$H_\sigma = \{(j,k) \;\mid\; \sigma_j \Rightarrow \sigma \text{ by using rule } r_k\}$$

the multiset of couple of indexes (j,k) such that the string σ is obtained from some string σ_j by application of some rule r_k. For every $\sigma \in M^{(t+1)}$ the truth value of the proposition "The string σ is in the region delimited by m_i" is the result of the following combination:

$$\mu_i(\sigma) = \overset{*}{\bigvee}_{(j,k) \in H_\sigma} \left(\mu_i(\sigma_j) \overset{\wedge}{*} \mu_i(r_k) \right).$$

The value $\mu_i(\sigma)$ is evaluated for each string σ and for all membranes m_i, in any configuration of the system. Hence, by considering the dynamical update of "how much" each string "belongs" to every membrane, we can determine the "communication" of strings, that is their movement across the vague boundaries of Π.

Let us display the multiset H_σ as $\{(j_1, k_1), (j_2, k_2), \ldots, (j_u, k_u)\}$. The formula defining $\mu_i(\sigma)$ can be read as follows:

$$\begin{array}{rll}
\text{EITHER} & \sigma \text{ is produced from } & \sigma_{j_1} \text{ by } r_{k_1} \\
\text{OR} & \sigma \text{ is produced from } & \sigma_{j_2} \text{ by } r_{k_2} \\
\vdots & & \\
\text{OR} & \sigma \text{ is produced from } & \sigma_{j_u} \text{ by } r_{k_u}
\end{array}$$

In the theory of fuzzy control the "connective" OR is naturally interpreted by a t-conorm, which generalizes the disjunctive character of classical Boolean disjunction in definition by cases. The t-norm used to combine the membership of σ_j with the membership of r_k generalizes the crisp concept of σ_j AND r_k belonging to the same membrane. Then it is worth examining the possible connections between our description of membrane systems with vague boundaries with the theory of t-norm based fuzzy control.

4 Discussion and future work

Here we collect some ideas for further discussion and future developments of our preliminary proposal:

1. Here we have considered P systems with vague boundaries where only string-objects are present inside the membrane structure. The use of structured objects takes inspiration from the biology of the cell, where long molecules (for instance, proteins) can live across the phospholipidic bilayer of membranes, thus having a part inside and another part outside the organelle delimited by that membrane. What about the natural extension of P systems with vague boundaries to the case of multisets of symbol-objects?

2. In a cell, many transmembrane proteins act as channels or gates for the (selective) passage of biochemical substances. In [1] the functioning of sodium-potassium exchange pump is modelled within the framework of P systems, and the notion of bilayer is defined in order to have a realistic description of the cellular process. Hence, it would be interesting to introduce the same notion of bilayer also in P systems with vague boundaries, and to define the membership values of objects and rules not only for all membranes but also for their corresponding bilayer.

3. It could be interesting to adapt our definition of membranes with vague boundaries to describe hierarchical systems where the notion of sphere of influence plays a key role. In this setting, the concepts of distributions of objects and their topological or metrical relationships could be modeled by adding structure, in the form of logical or analytical constraints, to our basic description.

References

[1] D. Besozzi and G. Ciobanu. A P system description of the sodium-potassium pump. *Submitted work*, 2004.

[2] P. Hájek. *Metamathematics of fuzzy logic.* Kluwer Academic Publishers, Dordrecht, 1998.

[3] E. P. Klement, R. Mesiar, and E. Pap. *Triangular Norms.* Kluwer Academic Publishers, Dordrecht, 2000.

[4] V. Novák, I. Perfilieva, and J. Močkoř. *Mathematical principles of fuzzy logic.* Kluwer, Dordrecht, 1999.

[5] Gh. Păun. Computing with membranes. *Journal of Computer and System Sciences*, 61(1):108–143, 2000. (See also *Turku Center for Computer Science-TUCS Report* No 208, 1998, www.tucs.fi).

[6] Gh. Păun. *Membrane Computing. An Introduction.* Springer-Verlag, Berlin, 2002.

[7] J. Yen and R. Langari. *Fuzzy Logic. Intelligence, Control, and Information.* Prentice-Hall, Upper Saddle River, 1999.

[8] L. Zadeh. Fuzzy sets. *Information and Control*, 8(3):338–353, 1965.

[9] L. Zadeh. Soft computing and fuzzy logic. *IEEE Software*, 11(6):48–56, 1994.