Counting the contents of fuzzy membranes... and related problems[1]

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The content of a membrane in a configuration of an 'exact' P system is described by a multiset. Recall that a (crisp) *multiset* over a set of *types* X is simply a mapping $d : X \to \mathbb{N}$. The usual interpretation of a multiset $d : X \to \mathbb{N}$ is that it describes a set consisting of $d(x)$ "exact" copies of each type $x \in X$. In particular, it is assumed that the set described by the multiset does not contain any element that is not a copy of some $x \in X$, or rather that we do not care about these elements, and that an element of it cannot be a copy of two different types.

Now, the uncertainty in an 'inexact' P system may arise at the level of the (lack of) crispness of its membranes' contents, and this can be represented using fuzzy multisets of different kinds. For instance, we could understand that the objects are imperfect, approximate copies of the reactives purportedly involved in its reactions. This would lead us to multisets describing, for every reactive v and for every degree of approximation t, how many elements there are in the membrane that are approximate copies of v with (or within) degree of approximation t. We could also understand that our lack of knowledge of the system refers to the number of copies of the (now, exact) reactives in each membrane. This would lead us to multisets describing, for every reactive v and for every $n \in \mathbb{N}$, the degree of certainty of there being n copies of v in the membrane. And so on.

Even using these generalized kinds of multisets, the basic processes of P systems based on them would be still removing, creating and moving objects within the system, and the final result of a computation would be still obtained by counting (in some way) the objects in some membrane. This calls for the development and study of *cardinalities* to 'count' the kind of fuzzy multisets used in this context.

We consider here two types of cardinalities: *scalar*, assigning to each multiset a positive real number, and *fuzzy*, assigning to each multiset a *fuzzy natural number*, a fuzzy subset of $\mathbb{N}$ with certain properties. Both may have their interest in different types of P systems. Fuzzy membrane systems with scalar cardinalities would produce computable (in the membrane sense) subsets of $\mathbb{R}^+$, while fuzzy membrane systems using fuzzy cardinalities would produce computable sets of fuzzy natural numbers.

The results on scalar and crisp cardinalities of fuzzy multisets of the first type discussed above (fuzzy multisets of approximate copies) contained in Sections 4 and 6 of this note were proved in [3]. The rest of this note is devoted to discuss work currently in progress. Previous studies of fuzzy cardinalities of fuzzy multisets include [1, 2, 4, 5]

1 Fuzzy natural numbers

A *generalized natural number* is a mapping $\nu : \mathbb{N} \to [0, 1]$; the set of generalized natural numbers is, then, $[0, 1]^{\mathbb{N}}$. The *support* of a generalized natural number $\nu : \mathbb{N} \to [0, 1]$ is the set

$$Supp(\nu) = \{n \in \mathbb{N} \mid \nu(n) > 0\}.$$

We can include $\mathbb{N}$ into $[0, 1]^{\mathbb{N}}$ in several ways. For instance:

- By associating to every $n \in \mathbb{N}$ the generalized natural number $\overline{n} : \mathbb{N} \to [0, 1]$ defined by $\overline{n}(n) = 1$ and $\overline{n}(m) = 0$ for every $m \neq n$.
- By associating to every $n \in \mathbb{N}$ the generalized natural number $\widehat{n} : \mathbb{N} \to [0, 1]$ defined by $\widehat{n}(m) = 1$ if $m \leqslant n$ and $\widehat{n}(m) = 0$ if $m > n$.
- By associating to every $n \in \mathbb{N}$ the generalized natural number $\widetilde{n} : \mathbb{N} \to [0, 1]$ defined by $\widetilde{n}(m) = 0$ if $m < n$ and $\widetilde{n}(m) = 1$ if $m \geqslant n$.

Notice that $\overline{0} = \widehat{0} \neq \widetilde{0}$ and that $\overline{n} = \widehat{n} \wedge \widetilde{n}$.

It has been agreed that the 'sum' of generalized natural numbers corresponds to the following operation $\oplus$ on $[0, 1]^{\mathbb{N}}$, called the *extended sum*: for every $\nu, \mu \in [0, 1]^{\mathbb{N}}$,

$$(\nu \oplus \mu)(k) = \bigvee \{\nu(i) \wedge \mu(j) \mid i + j = k\} \text{ for every } k \in \mathbb{N}.$$

This extended sum of generalized natural numbers is associative, commutative, its neutral element is $\overline{0}$, and it extends the sum of natural numbers for each one of the embeddings described above. Moreover, the extended sum of two increasing (resp., decreasing) generalized natural numbers is again increasing (resp., decreasing).

We shall not use all generalized natural numbers but a certain subset of them.

A generalized natural number is *convex* when $\nu(k) \geqslant \nu(i) \wedge \nu(j)$ for every $i \leqslant k \leqslant j$. By a *fuzzy natural number* we shall understand a convex generalized natural number.

We shall say that a fuzzy natural number has a *summit* when it takes its greatest value in, and only in, an element $n_0 \in \mathbb{N}$, and that it has a *plateau* when it takes its greatest value in, and only in, all elements of an interval $\{n_0, n_0 + 1, \ldots, n_0 + k\} \subseteq \mathbb{N}$. Every fuzzy natural number has a summit or a plateau, and it increases to the left of it and decreases to the right of it.

Every increasing or decreasing generalized natural number is convex, and the extended sum of two convex generalized natural numbers is again convex.

It would also be natural to impose the finiteness of the support of fuzzy natural numbers, which would entail that at some distance to the left and, specially, to the right of the summit of the generalized natural number, it is defined 0. We shall not do it here (mainly because the generalized natural numbers $\widetilde{n}$ do not have a finite support), although in some contexts this restriction appears in a natural way: see Section 4.

The extended sum of two fuzzy natural numbers (resp., with finite support) is again a fuzzy natural number (resp., with finite support).

We shall denote by $\overline{\mathbb{N}}$ the set of all fuzzy natural numbers. The embeddings $\mathbb{N} \hookrightarrow [0,1]^{\mathbb{N}}$ described above are embeddings $\mathbb{N} \hookrightarrow \overline{\mathbb{N}}$.

We shall use fuzzy natural numbers as models of 'imprecisely known' natural numbers, taking as 'exact natural numbers' the images of one of these embeddings $\mathbb{N} \hookrightarrow \overline{\mathbb{N}}$. So, for instance, our knowledge of a quantity that lies 'around 5' will be represented by a fuzzy natural number with a summit in 5, or a plateau around 5.

2 Fuzzy multisets of uncertain quantities

As we mentioned in the introduction, a natural definition of fuzzy multiset assigns to each element of a set of types an imprecisely known natural number. Since we are advocating here for the use of fuzzy natural numbers (actually, of some suitable subset of them; see the next section) as models of the latter, this leads us to the following definition.

Definition 1 *A* fuzzy multiset of uncertain quantities*, a* uq-fuzzy multiset *for short, over a set of* types X *is a mapping* $F : X \to \overline{\mathbb{N}}$. *Such a uq-fuzzy multiset is* finite *if its* support

$$Supp(F) = \{x \in X \mid F(x) \neq \overline{0}\}$$

is a finite subset of X.

For every uq-fuzzy multisets F, G over a set X, their *sum* is the uq-fuzzy multiset $A + B$ defined pointwise by

$$(A + B)(x) = A(x) \oplus B(x), \quad \text{for every } x \in X.$$

A *scalar cardinality* of uq-multisets would measure their size by means of positive real numbers, assigning moreover to each crisp multiset its usual cardinal. Such a scalar cardinality could be obtained by taking any morphism of monoids

$$\alpha : \overline{\mathbb{N}} \to \mathbb{R}^{+}$$

that preserves the chosen embedding $\mathbb{N} \hookrightarrow \overline{\mathbb{N}}$, and then defining

$$Sc_{\alpha}(F) = \sum_{x \in Supp(F)} \alpha(F(x)).$$

Actually, if we define abstractly a *scalar cardinality* of uq-fuzzy multisets as a mapping that sends every uq-fuzzy multiset to a positive real number, preserves the sums and extends the usual cardinality of crisp multisets, then it is not difficult to prove that all such scalar cardinalities are obtained in this way.

Anyway, the natural definition of the cardinal $\mathcal{C}$ of a finite uq-fuzzy multiset F over a set of types X assigns to each one of them a fuzzy natural number:

$$\mathcal{C}(F) = \bigoplus_{x \in Supp(F)} F(x) \in \overline{\mathbb{N}}.$$

This also extends the usual cardinality of crisp multisets.[2]

The main problem with uq-fuzzy multisets comes from a handicap of fuzzy natural numbers. In membrane processes, we must be able to compare and to subtract multisets. Now, the natural definition of $F \leqslant G$ for two uq-fuzzy multisets F and G should be

$$F(x) \leqslant G(x) \text{ in } \overline{\mathbb{N}}, \text{ for every } x \in X,$$

And when $F \leqslant G$, the natural definition of their difference $G - F$ should be

$$(G - F)(x) = G(x) - F(x) \text{ in } \overline{\mathbb{N}}, \text{ for every } x \in X.$$

But, what are these order and subtraction in $\overline{\mathbb{N}}$?

3 Subtracting fuzzy natural numbers

As we see, the use of fuzzy natural numbers to describe our imprecise knowledge of the number of reactives in a membrane at a given moment of a process poses two problems: comparison and subtraction. Given two fuzzy natural numbers μ and ν, if μ is larger than ν, how can we subtract ν from μ, finding a fuzzy natural number $\mu - \nu$ such that

$$\nu \oplus (\mu - \nu) = \mu?$$

Would it be uniquely determined?

And, actually, to begin with, what does 'larger' mean in $\overline{\mathbb{N}}$? There are some proposals in this connection [7]. All of them translate in some sense the intuitive idea that if $\nu \leqslant \mu$, then the 'increasing' and the 'decreasing' branches of ν should lie to the left of those of μ, respectively. But they do not yield a well-defined subtraction.

Then, if we want (and we want!) to describe uncertainly known quantities of reactives in a membrane as fuzzy natural numbers, we need to know how to subtract them.

First of all, notice that if the rules in the membrane system remove crisp quantities of reactives, then we only need to compare natural numbers, embedded in $\overline{\mathbb{N}}$ as we had decided to do it, with fuzzy natural numbers, and to subtract a natural number from a fuzzy natural number. Let's take a glance at the embeddings given a the beginning.

- Assume that we take the embedding $n \mapsto \overline{n}$. For every $\nu \in \overline{\mathbb{N}}$ and $n \in \mathbb{N}$, and for every $m \in \mathbb{N}$, we have that, if $\nu - \overline{n}$ is defined, then, for every $i = 0, \ldots, m$,

$$(\nu - \overline{n})(i) \wedge \overline{n}(m-i) = \begin{cases} 0 & \text{if } m - i \neq n, \text{ i.e., if } i \neq m - n \\ (\nu - \overline{n})(i) & \text{if } m - i = n, \text{ i.e., if } i = m - n \end{cases}$$

  which implies that

$$\bigvee_{i=0,\ldots,m} (\nu - \overline{n})(i) \wedge \overline{n}(m-i) = \begin{cases} 0 & \text{if } m < n \\ (\nu - \overline{n})(m-n) & \text{if } n \leqslant m \end{cases}$$

  In particular, if it has to be $\nu(m)$, we have that $\nu(m) = 0$ for every $m < n$. This leads us to the following definition-result:

[2] It is not difficult to define abstractly *fuzzy cardinality* of uq-fuzzy multisets and to characterize them as we do it for the fuzzy cardinalities of another type of fuzzy multisets in Section 6. We shall not do it here.

Proposition 1 *For every $\nu \in \overline{\mathbb{N}}$ and $n \in \mathbb{N}$, we define that*

$$\overline{n} \leqslant \nu \text{ if and only if } \nu(m) = 0 \text{ for every } m < n.$$

And if $\overline{n} \leqslant \nu$, then we define $\nu - \overline{n} \in \overline{\mathbb{N}}$ by $(\nu - \overline{n})(i) = \nu(n+i)$ for every $i \in \mathbb{N}$.

With these definitions, if $\overline{n} \leqslant \nu$, then the fuzzy natural number $\nu - \overline{n}$ is the only one such that $\overline{n} \oplus (\nu - \overline{n}) = \nu$.

- Assume now that we take the embedding $n \mapsto \widehat{n}$. In this case more involved discussion proves the following result.

  Proposition 2 *For every $\nu \in \overline{\mathbb{N}}$ and $n \geqslant 1$, there exists some $\nu - \widetilde{n} \in \overline{\mathbb{N}}$ such that $\widetilde{n} \oplus (\nu - \widetilde{n}) = \nu$ if and only if ν has a plateau of at least $n+1$ elements.*

  And when ν has such a plateau $\{n_0, \ldots, n_0 + k_0\}$, then taking $(\nu - \widetilde{n})(i) = \nu(i)$ for every $i < n_0 + k_0$ and $(\nu - \widetilde{n})(i) = \nu(i+n)$ for every $i \geqslant n_0 + k_0$ we obtain a fuzzy natural number such that $\widehat{n} \oplus (\nu - \widehat{n}) = \nu$, but not all of them.

  Thus, we can define

  $$\widehat{n} \leqslant \nu \text{ if and only if } \nu \text{ has a plateau of at least } n+1 \text{ elements,}$$

  and the subtraction $\nu - \widehat{n}$ as in the last proposition.

  Since the fuzzy natural numbers $\widehat{n}$ are decreasing, it is natural to use them when we only consider *decreasing* fuzzy natural numbers. For decreasing fuzzy numbers, the order defined above becomes

  $$\widehat{n} \leqslant \nu \text{ if and only if } \nu(0) = \cdots = \nu(n)$$

  and then, when $\widehat{n} \leqslant \nu$, taking $\nu - \widehat{n}$ defined by $(\nu - \widehat{n})(i) = \nu(i+n)$ for every $i \in \mathbb{N}$, yields a solution of $\widehat{n} \oplus (\nu - \widehat{n}) = \nu$.

- A similar situation happens with the embeddings $n \mapsto \widetilde{n}$. We leave the details to the reader.

In the general situation, if we want to remove uncertain quantities of reactives from a membrane where we have other uncertain quantities of reactives, we need to give some answer to the following question.

> **Open question.** To identify a meaningful and general enough subset $\overline{\mathbb{N}}'$ of $\overline{\mathbb{N}}$ where a meaningful (possibly partial) order $\leqslant$ can be defined in such a way that, for every μ, ν belonging to this subset, if $\nu \leqslant \mu$, then there exists one distinguished element $\mu - \nu$ such that $\nu \oplus (\mu - \nu) = \mu$. Moreover, $\mathbb{N}$ should be embedded into $\overline{\mathbb{N}}'$ in some way.

This would mean, of course, that we would allow the "uncertain quantities of reactives" to lie only in $\overline{\mathbb{N}}'$, i.e., to define *uq-fuzzy multisets* as mappings $X \to \overline{\mathbb{N}}'$.

For instance, if we restrict ourselves to *decreasing fuzzy natural numbers*, then the solution of the general problem is the following. We define

$$\nu \leqslant \mu \text{ if and only if } |\nu^{-1}(t)| \leqslant |\mu^{-1}(t)|, \text{ for every } t \in]0,1],$$

and then, a subtraction satisfying the desired property can be defined as follows: if $\nu \leqslant \mu$ in this sense, then we consider the mapping $H(\mu,\nu) :]0,1] \to \mathbb{N}$ defined by

$$H(\mu,\nu)(t) = |\mu^{-1}(t)| - |\nu^{-1}(t)| \quad \text{for every } t \in]0,1],$$

and then

$$(\mu - \nu)(n) = \bigvee \{t \in [0,1] \mid \sum_{t' \geqslant t} H(\mu,\nu)(t') \geqslant n\}.$$

This is a decreasing fuzzy natural number such that $\mu \oplus (\nu - \mu) = \nu$, but it is the only one.

These order and subtraction were first described by A. Obtułowicz in [6], and can be obtained as a slight modification of a particular case of a general construction that we shall discuss in Section 5.

If $\nu = \widehat{n}$, for some n, then, with this order, $\widehat{n} \leqslant \mu$ if and only if $\mu(0) = \cdots = \mu(n) = 1$, and then the subtraction agrees with the one described above in the particular case of subtracting $\widehat{n}$ from decreasing fuzzy natural numbers.

If we restrict ourselves to *increasing fuzzy natural numbers*, then our problem also has a solution. The order and the corresponding subtraction can be obtained again as a particular case of the aforementioned general construction we shall give later.

In general, in Section 5 we shall show a method to produce families of fuzzy natural numbers where an order and a subtraction can be defined. We do not know whether some of them is natural enough to be used in practice: perhaps a nice subfamily of one of them will work. The interesting fact is that our families arise as another type of fuzzy cardinalities of multisets.

4 Fuzzy cardinalities of finite multisets on $]0,1]$.

A (*crisp*) *multiset* over $]0,1]$ is a mapping $A :]0,1] \to \mathbb{N}$. A multiset A over $]0,1]$ is *finite* if its *support*

$$Supp(A) = \{t \in]0,1] \mid A(t) > 0\}$$

is a finite subset of $]0,1]$. We shall denote the set of all finite multisets over $]0,1]$ by $FMS(]0,1])$, and by $\perp$ the *null multiset*, defined by $\perp(t) = 0$ for every $t \in]0,1]$.

For every $A, B \in FMS(]0,1])$, their *sum* $A + B$ is the multiset

$$(A+B)(t) = A(t) + B(t), \qquad \text{for every } t \in]0,1].$$

We shall denote by n/t the multiset sending t to n and every $t' \neq t$ to 0.

A fuzzy cardinality of a finite multiset A over $]0,1]$ is a fuzzy natural number that measures how many elements has A.

Definition 2 *A* fuzzy cardinality *on $FMS(]0,1])$ is a mapping $\mathcal{C} : FMS(]0,1]) \to \overline{\mathbb{N}}$ that satisfies the following conditions:*

(i) For every $A, B \in FMS(]0,1])$, $\mathcal{C}(A+B) = \mathcal{C}(A) \oplus \mathcal{C}(B)$.

(ii) For every $A, B \in FMS(]0,1])$ and for every $i, j \in \mathbb{N}$ such that $i > \sum_{t \in Supp(A)}(A)$ and $j > \sum_{t \in Supp(A)}(B)$, $\mathcal{C}(A)(i) = \mathcal{C}(B)(j)$.

(iii) If $Supp(A) \subseteq \{1\}$, then $\mathcal{C}(A)(i) \in \{0,1\}$ for every $i \in \mathbb{N}$ and, moreover, if $n = A(1)$, then $\mathcal{C}(A)(n) = 1$.

(iv) If $t, t' \in]0,1]$ are such that $t \leqslant t'$, then

$$\mathcal{C}(1/t)(0) \geqslant \mathcal{C}(1/t')(0) \quad \text{and} \quad \mathcal{C}(\perp)(1) \leqslant \mathcal{C}(1/t)(1) \leqslant \mathcal{C}(1/t')(1).$$

Let us explain the meaning as well as some motivations for each one of these conditions:

- Condition (i), *additivity*, generalizes to fuzzy natural numbers the additivity of the classical cardinal of a crisp multiset.
- Condition (ii) implements the idea that the elements t not belonging to the support of a finite multiset A should not affect the cardinality of A.
- Condition (iii) requires that, on each multiset of the form $n/1$, with $n \in \mathbb{N}$, any fuzzy cardinality must take values only in $\{0,1\}$, and the value 1 on the specific number n. If in this property we restrict the type of cardinalities we accept for $n/1$, then the overall set of cardinalities is restricted.
- Condition (iv) captures the restriction that the value of the cardinality of singletons on 0 must decrease and their value on 1 must increase with the element of their support.

The bracket fuzzy cardinality defined in the next example will play a key role henceforth.

Example 3 *Let us consider the function*

$$\begin{array}{rrcl} [\ ]: & FMS(]0,1]) & \to & [0,1]^{\mathbb{N}} \\ & A & \mapsto & [A] \end{array}$$

where, for every $A \in FMS(]0,1])$,

$$\begin{array}{rrcl} [A]: & \mathbb{N} & \to & [0,1] \\ & i & \mapsto & [A]_i \end{array}$$

is defined by

$$[A]_i = \bigvee \{t \in [0,1] \mid \sum_{t' \geqslant t} A(t') \geqslant i\}.$$

It is clear that $[A]$ *is decreasing and that* $[A]_i = 0$ *for every* $i > \sum_{t \in Supp(A)}(A)$*, and hence* $[A] \in \overline{\mathbb{N}}$ *for every* $A \in FMS(]0,1])$*. It turns out that this mapping* $A \mapsto [A]$ *is a fuzzy cardinality on* $FMS(]0,1])$*, which we shall call the* bracket *cardinality.*

Definition 3 *Let* $f : [0,1] \to [0,1]$ *be an increasing mapping such that* $f(0) \in \{0,1\}$ *and* $f(1) = 1$ *and let* $g : [0,1] \to [0,1]$ *be a decreasing mapping such that* $g(0) = 1$ *and* $g(1) \in \{0,1\}$*.*

Let $\mathcal{C}_{f,g} : FMS(]0,1]) \to \overline{\mathbb{N}}$ *be the mapping defined as follows: for every* $A \in FMS(]0,1])$ *and* $i \in \mathbb{N}$,

$$\mathcal{C}_{f,g}(A)(i) = f([A]_i) \wedge g([A]_{i+1}).$$

The key theorem in this section is the following.

Theorem 4 *A mapping* $\mathcal{C} : FMS(]0,1]) \to \overline{\mathbb{N}}$ *is a fuzzy cardinality if and only if* $\mathcal{C} = \mathcal{C}_{f,g}$ *for some increasing mapping* $f : [0,1] \to [0,1]$ *such that* $f(0) \in \{0,1\}$ *and* $f(1) = 1$ *and some decreasing mapping* $g : [0,1] \to [0,1]$ *such that* $g(0) = 1$ *and* $g(1) \in \{0,1\}$*.*

The last theorem allows us to call the mapping $\mathcal{C}_{f,g}$, for every f, g as in Definition 3, the *fuzzy cardinality generated by* f *and* g. It provides an explicit description of all fuzzy cardinalities in terms of the bracket cardinality.

Proposition 5 $\mathcal{C}_{f,g}(A)$ *is increasing for every* $A \in FMS(]0,1])$ *if and only if* f *is the constant mapping* 1*, in which case* $\mathcal{C}_{f,g}(A)(k) = g([A]_{k+1})$ *for every* $A \in FMS(]0,1])$ *and* $k \in \mathbb{N}$*.*

Proposition 6 $\mathcal{C}_{f,g}(A)$ *is decreasing for every* $A \in FMS(]0,1])$ *if and only if* g *is the constant mapping 1, in which case* $\mathcal{C}_{f,g}(A)(k) = f([A]_k)$ *for every* $A \in FMS(]0,1])$ *and* $k \in \mathbb{N}$*.*

Since, for every f, g as in Definition 3 and, for every $A \in FMS(]0,1])$ and $k \in \mathbb{N}$,

$$\mathcal{C}_{f,g}(A)(k) = f([A]_k) \wedge g([A]_{k+1}),$$

we deduce the following result.

Corollary 7 *Every fuzzy cardinality on* $FMS(]0,1])$ *is the meet of an increasing fuzzy cardinality and a decreasing fuzzy cardinality: namely* $\mathcal{C}_{f,g} = \mathcal{C}_{f,1} \wedge \mathcal{C}_{1,g}$*.*

The equality in the last statement and the fact that, for every A, $\mathcal{C}_{f,1}(A)$ is decreasing and $\mathcal{C}_{1,g}(A)$ is increasing, easily entail that, in the non-trivial cases when neither f nor g are the constant mapping 1, there exists an $n_0 \in \mathbb{N}$ such that

$$\mathcal{C}_{f,g}(A)(i) = \begin{cases} \mathcal{C}_{1,g}(A)(i) & \text{if } i < n_0 \\ \mathcal{C}_{f,1}(A)(i) & \text{if } i \geqslant n_0 \end{cases}$$

These give the increasing and decreasing branches of $\mathcal{C}_{f,g}(A)$.

5 Subtracting fuzzy natural numbers revisited

Let $\leqslant$ denote the partial order on $FMS(]0,1])$ defined pointwise by

$$A \leqslant B \text{ if and only if } A(t) \leqslant B(t) \text{ for every } t \in]0,1].$$

If $A \leqslant B$, then their *difference* $B - A$ is the multiset defined pointwise by

$$(B - A)(t) = B(t) - A(t) \text{ for every } t \in]0,1].$$

Proposition 8 *Let* $\mathcal{C}$ *be a fuzzy cardinality on* $FMS(]0,1])$*. If* $A, B \in FMS(]0,1])$ *are such that* $A \leqslant B$*, then*

$$\mathcal{C}(A) \oplus \mathcal{C}(B - A) = \mathcal{C}(B).$$

This makes us return to the open question that we posed in Section 3. In view of Proposition 8, a possible answer to it would be to take, for any fuzzy cardinality $\mathcal{C}$ on $FMS(]0,1])$,

$$\mathbb{N}_{\mathcal{C}} = \{\mathcal{C}(A) \in \overline{\mathbb{N}} \mid A \in FMS(]0,1])\},$$

to define on this set the partial order

> $\nu \preccurlyeq \mu$ if and only if there exist $A, B \in FMS(]0,1])$ such that $\nu = \mathcal{C}(A)$, $\mu = \mathcal{C}(B)$ and $A \leqslant B$,

and then to define, for every $A, B \in FMS(]0,1])$ such that $A \leqslant B$,

$$\mathcal{C}(B) - \mathcal{C}(A) = \mathcal{C}(B - A).$$

This poses, of course, several technical questions. Is $\preccurlyeq$ a wel-defined partial order? Is the subtraction in $\mathbb{N}_{\mathcal{C}}$ well-defined, in the sense that if $A, A', B, B' \in FMS(]0,1])$ are such that $\mathcal{C}(A) = \mathcal{C}(A')$, $\mathcal{C}(B) = \mathcal{C}(B')$, $A \leqslant B$, and $A' \leqslant B'$, does it always happen that

$$\mathcal{C}(B' - A') = \mathcal{C}(B - A)?$$

We conjecture that the answer is in general positive, but we have not been able to prove it. Anyway, we have the following result.

Proposition 9 *Let f, g be mappings as in Definition 3. If f and g are injective, then, for every $A, B \in FMS(]0,1])$, if $\mathcal{C}_{f,g}(A) = \mathcal{C}_{f,g}(B)$, then $A = B$.*

Thus, if we restrict the set of cardinalities to those generated by *bijective* mappings f and g, then the answers are indeed positive (although we still do not know whether $\mathcal{C}(B - A)$ is the only fuzzy natural number whose extended sum with $\mathcal{C}(A)$ yields $\mathcal{C}(B)$).

The third question is the characterization of the sets $\mathbb{N}_{\mathcal{C}}$. In this connection, we have the following results.

Proposition 10 *For every $\nu \in \overline{\mathbb{N}}$ there always exist injective mappings f, g as in Definition 3 such that $\nu \in \overline{\mathbb{N}}_{\mathcal{C}_{f,g}}$. But, given $\mu, \nu \in \overline{\mathbb{N}}$, there need not exist a fuzzy cardinality $\mathcal{C}$ such that $\mu, \nu \in \overline{\mathbb{N}}_{\mathcal{C}}$.*

Theorem 11 *Let f, g be two bijective mappings as in Definition 3, and let $t_0 \in [0,1]$ be the only point where they cross, i.e., such that $f^{-1}(t_0) = g^{-1}(t_0)$.*

For every $\nu \in \overline{\mathbb{N}}$, we have that $\nu \in \overline{\mathbb{N}}_{\mathcal{C}_{f,g}}$ if and only if one of the following two conditions holds:

(a) $\nu(n) \leqslant t_0$ for every $n \in \mathbb{N}$, and ν has a plateau.

(b) There is only one point n_0 such that $\nu(n_0) > t_0$, and then the values of $\nu(n_0 - 1)$, $\nu(n_0)$ and $\nu(n_0 + 1)$ are linked through some specific conditions (namely, there exist $t_1 < t_0 < t_2$ such that $\nu(n_0 - 1) = g(t_2)$, $\nu(n_0 + 1) = f(t_1)$ and $\nu(n_0) = f(t_2) \wedge g(t_1)$).

Besides cardinalities $\mathcal{C}_{f,g}$ generated by bijective mappings, we could also take $\mathcal{C}$ to be the bracket cardinality, or the increasing cardinality $A \mapsto 1 - [A]_{i+1}$. The first one yields all decreasing fuzzy multisets μ with finite support and $\mu(0) = 1$, while the second one yields all increasing fuzzy multisets with finite support. In this cases, the subtraction is also well defined through the construction provided above.

6 Fuzzy multisets of approximate copies

Let us consider now fuzzy multisets describing sets containing approximate copies of the types. In a first approach, by such a *fuzzy multiset* over a set of *types* X we would understand a mapping $\overline{A} : X \times [0,1] \to \mathbb{N}$. Such a fuzzy multiset would be understood to describe a set consisting of, for each $x \in X$ and for every $t \in [0,1]$, $\overline{A}(x,t)$ copies of x with degree of similarity t to it.

We shall impose two restrictions on this interpretation of a fuzzy multiset. First, the set described by the fuzzy multiset does not contain any element that is not a copy of some $x \in X$ with some non-negative degree of similarity —or rather, we do not care about them. This is a natural condition. Second, we assume that if an element of the set is an inexact copy of x with degree of similarity $t > 0$, then it cannot be an inexact copy of any other type in X with a non-negative degree of similarity. This is a strong condition, and we shall return on it at the end of this section. These two conditions entail that, for every $x \in X$, the value $\overline{A}(x,0)$ must be equal to $\sum_{w\in X-\{0\}} \sum_{t>0} \overline{A}(w,t)$ and in particular that the restriction of $\overline{A}$ to $X \times \{0\}$ is determined by the restriction of $\overline{A}$ to $X \times]0,1]$. This leads us finally to the following definition.

Definition 4 *A* fuzzy multiset of approximate copies*, an* ac-fuzzy multiset *for short, over a set X is a mapping $\overline{A} : X \times]0,1] \to \mathbb{N}$,, i.e., a mapping*

$$\overline{A} : X \times MS(]0,1]).$$

Such an ac-fuzzy multiset is finite *if its* support

$$Supp(\overline{A}) = \{x \in X \mid \overline{A}(x) \neq \bot\}$$

is a finite subset of X and, for every $x \in Supp(\overline{A})$, $\overline{A}(x) \in FMS(]0,1])$.

We shall denote the sets of all ac-fuzzy multisets and of all finite fuzzy multisets over X by $\mathcal{FMS}(X)$ and $\mathcal{FFMS}(X)$, respectively.

Given two ac-fuzzy multisets $\overline{A}, \overline{B}$ over X, their *sum* $\overline{A} + \overline{B}$ is the ac-fuzzy multiset over X defined pointwise by

$$(\overline{A} + \overline{B})(x) = \overline{A}(x) + \overline{B}(x) \text{ for every } x \in X.$$

For every $x \in X$ and $A \in MS(]0,1])$, we shall denote by A/x the fuzzy multiset over X defined by $(A/x)(x) = A$ and $(A/x)(y) = \bot$ for every $y \neq x$. Notice that if A is finite, then A/x is also finite, and that, for every $\overline{A} \in \mathcal{FFMS}(X)$,

$$\overline{A} = \sum_{x \in Supp(\overline{A})} (\overline{A}(x))/x.$$

The partial order $\leqslant$ on $\mathcal{FMS}(X)$ is defined by

$$\overline{A} \leqslant \overline{B} \text{ if and only if } \overline{A}(x) \leqslant \overline{B}(x) \text{ for every } x \in X.$$

If $\overline{A} \leqslant \overline{B}$, then their *difference* $\overline{B} - \overline{A}$ is the fuzzy multiset defined pointwise by

$$(\overline{B} - \overline{A})(x) = \overline{B}(x) - \overline{A}(x) \text{ for every } x \in X.$$

The scalar cardinality of a finite fuzzy multiset $\overline{A}$ is a real number that measures the overall size of the set described by $\overline{A}$.

Definition 5 *A* scalar cardinality *on* $\mathcal{FFMS}(X)$ *is a mapping* $Sc : \mathcal{FFMS}(X) \to \mathbb{R}^+$ *that satisfies the following conditions:*

(i) $Sc(\overline{A} + \overline{B}) = Sc(\overline{A}) + Sc(\overline{B})$ *for every* $\overline{A}, \overline{B} \in \mathcal{FFMS}(X)$.

(ii) $Sc((1/1)/x) = 1$ *for every* $x \in X$.

A scalar cardinality Sc *on* $\mathcal{FFMS}(X)$ *is* homogeneous *when it satisfies the following extra property:*

(iii) $Sc(M/x) = Sc(M/y)$ *for every* $x, y \in X$ *and* $M \in FMS(]0,1])$.

Next proposition provides a description of all scalar cardinalities on $\mathcal{FFMS}(X)$.

Proposition 12 *A mapping* $Sc : \mathcal{FFMS}(X) \to \mathbb{R}^+$ *is a scalar cardinality if and only if for every* $x \in X$ *there exists a mapping* $f_x :]0,1] \to \mathbb{R}^+$ *with* $f_x(1) = 1$*, such that, for every fuzzy multiset* $\overline{A}$ *over* X,

$$Sc(\overline{A}) = \sum_{x \in X} \sum_{t \in Supp(\overline{A}(x))} f_x(t)\overline{A}(x)(t).$$

Moreover, the mappings $(f_x)_{x \in X}$ *are uniquely determined by* Sc*, and* Sc *is homogeneous if and only if* $f_x = f_y$ *for every* $x, y \in X$.

Now, a fuzzy cardinality of a fuzzy multiset measures the size of the set it describes by means of a fuzzy natural number.

Definition 6 *A* fuzzy cardinality *on* $\mathcal{FFMS}(X)$ *is a mapping* $\mathcal{C} : \mathcal{FFMS}(X) \to \overline{\mathbb{N}}$ *that satisfies the following conditions:*

(i) For every $\overline{A}, \overline{B} \in \mathcal{FFMS}(X)$, $\mathcal{C}(\overline{A} + \overline{B}) = \mathcal{C}(\overline{A}) \oplus \mathcal{C}(\overline{B})$.

(ii) For every $x \in X$*, the mapping*

$$\begin{array}{rrcl} \mathcal{C}(\ /x): & FMS(]0,1]) & \to & \overline{\mathbb{N}} \\ & M & \mapsto & \mathcal{C}(M/x) \end{array}$$

is a fuzzy cardinality on $FM(]0,1])$

A fuzzy cardinality is homogeneous *when it satisfies the following further condition:*

(iii) For every $x, y \in X$, $\mathcal{C}(\ /x) = \mathcal{C}(\ /y)$.

Proposition 13 *A mapping* $\mathcal{C} : \mathcal{FFMS}(X) \to \overline{\mathbb{N}}$ *is a fuzzy cardinality if and only if for every* $x \in X$ *there exists an fuzzy cardinality* $\mathcal{C}_x : FMS(]0,1]) \to \overline{\mathbb{N}}$ *such that*

$$\mathcal{C}(\overline{M}) = \bigoplus_{x \in X} \mathcal{C}_x(\overline{M}(x)).$$

Moreover, the family $(\mathcal{C}_x)_{x \in X}$ *is uniquely determined by* $\mathcal{C}$*, and* $\mathcal{C}$ *is homogeneous if and only if* $\mathcal{C}_x = \mathcal{C}_y$ *for every* $x, y \in X$.

Thus, homogeneous scalar and fuzzy cardinalities understand fuzzy multisets as a sum of crisp multisets, one on every type $x \in X$, and "count" this sum. Arbitrary scalar and fuzzy cardinalities "count" each multiset on each $x \in X$, possibly using a different cardinality for every $x \in X$, and then add up these results.

Adding a fuzzy multiset to a fuzzy multiset corresponds to the extended sum of their cardinalities. As far a removing a fuzzy multiset from another fuzzy multiset, we still have the following result.

Corollary 14 *Let $\mathcal{C}$ be a fuzzy cardinality on $\mathcal{FFMS}(X)$. If $\overline{A}, \overline{B} \in \mathcal{FFMS}(X)$ are such that $A \leqslant B$, then*

$$\mathcal{C}(\overline{A}) \oplus \mathcal{C}(\overline{B} - \overline{A}) = \mathcal{C}(\overline{B}).$$

Therefore, the fuzzy cardinal of $\overline{B} - \overline{A}$ can be seen as the subtraction of the cardinal of $\overline{A}$ to that of $\overline{B}$. Notice anyway that now we are subtracting fuzzy multisets, and the subtraction of cardinals is a consequence of this operation.

We should remove our working hypothesis that an object can only be similar to only one reactive in order to cover more general situations, where objects can be similar to different reactives or even where reactives can be similar themselves. This would affect our constructions in two ways. The first one is that the values of a fuzzy multiset over a set X on elements of the form $(x, 0)$ would no longer be entailed by the rest of values, and thus multisets would have to be defined as mappings

$$F : X \to MS([0, 1]).$$

This is conceptually easy, although technically involved, to cope with.

But if we remove this hypothesis, then the order for fuzzy multisets and their subtractions become something darker. It is not the same to have an object similar to x and to y than two objects, one similar to x and the other similar to y: in the first case, when we remove one single object we get the null multiset, in the second case, not.

Our results on cardinalities of ac-fuzzy multisets deal with abstract objects and therefore they are formally correct in this new setting, but they are not sound. For instance, Proposition 14 is a direct consequence of additivity, but it should not held in this setting. Thus, cardinals of these fuzzy multisets would have to be handled in a completely different way, and uncertain P systems with membranes' contents described by these multisets would be more difficult to define.

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